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HOMOGENEOUS LINEAR SUBSTITUTIONS

BY

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PREFACE

IN this book I have attempted to put together for the benefit of the mathematical student those properties of the Homogeneous Linear Substitution with real or complex coefficients of which frequent use is made in the Theory of Groups and in the Theory of Bilinear Forms and Invariant-factors; but which have not hitherto been collected in a single treatise, so far as I am aware.

I have confined myself to those properties of substitutions which do not depend on any 'group' of such substitutions. The main reason for this limitation is the fact that the most interesting properties of substitution-groups may be found in the second edition of Prof. Burnside's *Theory of Groups* (Cambridge Univ. Press, 1911); and it does not seem worth while to repeat work already so excellently performed. I have also limited myself in the main to the discussion of substitutions whose determinant does not vanish.

In the range to which I have confined myself I have made considerable use of Dr. Bromwich's *Quadratic Forms* (Cambridge Tracts, No. 3, 1906), Prof. Bôcher's *Introduction to Higher Algebra* (Macmillan, 1907), Dr. Muth's *Theorie und Anwendung der*

Elementartheiler (Teubner, 1899), and Prof. Weber's *Lehrbuch der Algebra* (Braunschweig, 1898), in addition to numerous articles in mathematical periodicals. The reader will find that the methods of proof are in many cases new, though the results are for the most part well known.

I have made no attempt to insert complete references, though a reference is occasionally given if likely to prove useful to the reader.

The student will find it an assistance to the understanding of the subject if he will work through some of the examples, many of which are quite easy. I have given indications of the method of proof in most cases.

The reader who is more interested in the applications than in the abstract theory of the subject may confine his attention to the first four chapters in the first instance.

My best thanks are due to the Delegates of the Oxford University Press for kindly undertaking the publication of this book, and to the Staff of the Press for the care and skill with which the printing has been done.

H. H.

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CHAPTER I

ELEMENTARY PROPERTIES OF SUBSTITUTIONS

§ 1. Definition of Substitution.

SUPPOSE we are given the equations

$$\left.\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1m}x_m \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2m}x_m \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3m}x_m \\ &\vdots \\ x_m' &= a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mm}x_m \end{aligned}\right\} \dots\dots\dots \text{(i)}$$

Then the operation of replacing x_1 by x_1', x_2 by x_2', ..., x_m by x_m' is called the *homogeneous linear substitution* A of *degree* m defined by the equations (i).

In this book 'substitution' will always mean 'homogeneous linear substitution'.

The m quantities $x_1, x_2, \dots, x_m$ are called the *variables* of the substitution A; the quantities a_{ij} are called the *coefficients*. Both variables and coefficients are ordinary real or complex quantities.

The array

$$\begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & & \\ & \cdot & \cdot & & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mm} \end{vmatrix}$$

will be called the *matrix* of the substitution A. If the matrix is considered as a determinant, it is called the *determinant* of A. We shall suppose that this determinant does not vanish, unless the contrary is stated.

The substitution A will often be denoted by

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \dots + a_{tm}x_m \;(t = 1, 2, \dots, m)$$

or by

$$(a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m,\; a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m, \\ \dots,\; a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mm}x_m).$$

The substitution

$$x_1' = x_1, x_2' = x_2, \ldots, x_m' = x_m$$

is called the *unit substitution*. It will be denoted by the symbol E.*

Ex. 1. If $f(x, y) = 0$ is the equation of a curve referred to rectangular Cartesian axes, the equation referred to the axes obtained by turning the original axes about the origin through an angle ω is found by performing on $f(x, y) = 0$ the substitution $(x \cos\omega - y \sin\omega,\ x \sin\omega + y \cos\omega)$.

What is the similar result in three dimensions?

Ex. 2. The product xy is reduced to the difference of two squares by performing on it the substitution $(x+y,\ x-y)$.

§ 2. Products.

Given two substitutions A and B

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \; (t = 1, 2, \ldots, m)$$

and

$$x_t' = b_{t1}x_1 + b_{t2}x_2 + \ldots + b_{tm}x_m \; (t = 1, 2, \ldots, m);$$

then P is called the *product* of A and B, if P is the substitution

$$x_t' = p_{t1}x_1 + p_{t2}x_2 + \ldots + p_{tm}x_m \; (t = 1, 2, \ldots, m),$$

where

$$p_{ij} = b_{i1}a_{1j} + b_{i2}a_{2j} + \ldots + b_{im}a_{mj} \; (i, j = 1, 2, \ldots, m) \ldots \text{(i)}$$

This is denoted symbolically by the equation $P = A \,.\, B$ or $P = AB$.†

We can obtain P conveniently in practice by operating with A on the right-hand side of the equations

of B.

$$x_t' = b_{t1}x_1 + b_{t2}x_2 + \ldots + b_{tm}x_m$$

We thus obtain

$$x_t' = b_{t1}(a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m) + b_{t2}(a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m) + \ldots + b_{tm}(a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mm}x_m),$$

or

$$x_t' = p_{t1}x_1 + p_{t2}x_2 + \ldots + p_{tm}x_m$$

from equations (i).

We see immediately that AB and BA are not in general the same substitution. If they are, so that $AB = BA$, A and B are said to be *permutable* or *commutative*. The conditions for this are by (i)

$$b_{i1}a_{1j} + b_{i2}a_{2j} + \ldots + b_{im}a_{mj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{im}b_{mj} \quad (i, j = 1, 2, \ldots, m).$$

* From the German 'Einheit'. Bôcher in his 'Higher Algebra' uses I.

† The definition of 'product' holds good, even if the determinant of A or B is zero.

We readily prove $AE = EA = A$.

By the ordinary law for multiplying determinants we have, using equations (i),

$$\begin{vmatrix} p_{11} & p_{12} & \cdot & \cdot & \cdot & p_{1m} \\ p_{21} & p_{22} & \cdot & \cdot & \cdot & p_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{m1} & p_{m2} & \cdot & \cdot & \cdot & p_{mm} \end{vmatrix} = \begin{vmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1m} \\ b_{21} & b_{22} & \cdot & \cdot & \cdot & b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{m1} & b_{m2} & \cdot & \cdot & \cdot & b_{mm} \end{vmatrix}$$

$$\times \begin{vmatrix} a_{11} & a_{21} & \cdot & \cdot & \cdot & a_{m1} \\ a_{12} & a_{22} & \cdot & \cdot & \cdot & a_{m2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{1m} & a_{2m} & \cdot & \cdot & \cdot & a_{mm} \end{vmatrix}.$$

Therefore :—

The determinant of the product of two substitutions is the product of their determinants.

Suppose now that C is

$$x_t' = c_{t1}x_1 + c_{t2}x_2 + \ldots + c_{tm}x_m \; (t = 1, 2, \ldots, m)$$

and that P is the product of AB and C. Then

$$\begin{aligned} p_{ij} = c_{i1}(b_{11}a_{1j} + b_{12}a_{2j} + \ldots + b_{1m}a_{mj}) \\ + c_{i2}(b_{21}a_{1j} + b_{22}a_{2j} + \ldots + b_{2m}a_{mj}) + \ldots \\ + c_{im}(b_{m1}a_{1j} + b_{m2}a_{2j} + \ldots + b_{mm}a_{mj}), \end{aligned}$$

or

$$p_{ij} = \Sigma c_{i\gamma} b_{\gamma\beta} a_{\beta j} \; (\gamma, \beta = 1, 2, \ldots, m) \text{............ (ii)}$$

Forming the product of A and BC we get the same substitution; i.e. $AB \,.\, C = A \,.\, BC = P$.

We say that P is 'the product ABC of A, B, and C'.

Similarly $ABC \,.\, D = AB \,.\, CD = A \,.\, BCD$; and so on for the product of any number of substitutions.

The reader will have no difficulty in proving by induction that, if $P = ABC \ldots KLM$, then

$$p_{ij} = \Sigma m_{i\mu} l_{\mu\lambda} k_{\lambda\kappa} \ldots c_{\delta\gamma} b_{\gamma\beta} a_{\beta j} \; (\mu, \lambda, \kappa, \ldots, \delta, \gamma, \beta = 1, 2, \ldots, m);$$

and that the determinant of the product of any number of substitutions is equal to the product of their determinants.

Ex. 1. Find the product of $x' = x - y + z$, $y' = -x + y + 2z$, $z' = 4x - 2y - z$ and of $x' = 2x + 2y - 3z$, $y' = y - z$, $z' = x + y + z$.

[In the second of these, substitute from the first $x - y + z$ for x, $-x + y + 2z$ for y, $4x - 2y - z$ for z.

We get
$$\left.\begin{aligned} x' &= 2(x-y+z)+2(-x+y+2z)-3(4x-2y-z)\\ y' &= \qquad\qquad\quad (-x+y+2z)-\ (4x-2y-z)\\ z' &= (x-y+z)\ +\ (-x+y+2z)+\ (4x-2y-z)\end{aligned}\right\},$$
or $x' = -12x+6y+9z,\ y' = -5x+3y+3z,\ z' = 4x-2y+2z,$
which is the required substitution.]

Ex. 2. If $A\equiv(2x+y, 3x+2y),\quad B\equiv(5x-2y,\ 2x-y),$
$$C\equiv(x-y,\ x+y),$$
find ABC, BCA, CAB.

[$ABC\equiv(3x+y,\ 5x+y)$, $BCA\equiv(13x-5y,\ 23x-9y)$.
$CAB\equiv(5x-3y,\ x-y)$.]

Ex. 3. If $A\equiv(x-y+2z,\ 3x-y,\ -2x-y+z)$
and $B\equiv(4x-2y+2z,\ y+z,\ y)$,
find AB and BA.

[$(-6x-4y+10z,\ x-2y+z.\ 3x-y)$ and
$(4x-y+z,\ 12x-7y+5z,\ -8x+4y-5z)$.]

Ex. 4. $(x+y+z,\quad -x+y-w,\quad x+z+w,\quad -y-z+w)$ and $(2x-y+z+w,\quad 2y+3z-w,\quad 2x-y-2z+3w,\quad -3x-2z-2w)$ are permutable.

[Their product is $(4x+2z+3w,\ x+3y+4z,\ x-2y-3z+2w,$
$-5x-y-3z-4w)$.]

Ex. 5. $(ax+by,\ cx+dy)$ and $(Ax+By,\ Cx+Dy)$ are permutable if $a-d:b:c=A-D:B:C$.

Ex. 6. The substitutions with matrices
$$\begin{vmatrix} a & b & c & f & g\\ 0 & a & b & 0 & f\\ 0 & 0 & a & 0 & 0\\ 0 & h & i & d & e\\ 0 & 0 & h & 0 & d\end{vmatrix} \text{ and } \begin{vmatrix} \alpha & 1 & 0 & 0 & 0\\ 0 & \alpha & 1 & 0 & 0\\ 0 & 0 & \alpha & 0 & 0\\ 0 & 0 & 0 & \alpha & 1\\ 0 & 0 & 0 & 0 & \alpha\end{vmatrix} \text{ are permutable.}$$

Ex. 7. A substitution with matrix of the form
$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_m\\ \alpha_m & \alpha_1 & \alpha_2 & \cdots & \alpha_{m-1}\\ \alpha_{m-1} & \alpha_m & \alpha_1 & \cdots & \alpha_{m-2}\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_1\end{vmatrix} \text{ or } \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_m\\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_1\\ \alpha_3 & \alpha_4 & \alpha_5 & \cdots & \alpha_2\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ \alpha_m & \alpha_1 & \alpha_2 & \cdots & \alpha_{m-1}\end{vmatrix}$$
is called 'cyclant of type I' or 'cyclant of type II' respectively. Show that the product of any two cyclant substitutions of type I is of type I; that the product of any two cyclant substitutions of type II is of type I; and that the product of a cyclant substitution of type I and a cyclant substitution of type II is of type II.

Show that any two substitutions of type I are permutable; but that this is not in general true of type II.

Ex. 8. If $S_1 \equiv (a_1x_1+b_1x_2+c_1x_3+d_1x_4+\ldots,$
$a_1x_2+b_1x_3+c_1x_4+\ldots,\ a_1x_3+b_1x_4+\ldots,\ \ldots,\ a_1x_m),$
$S_2 \equiv (a_2x_1+b_2x_2+c_2x_3+d_2x_4+\ldots,$
$a_2x_2+b_2x_3+c_2x_4+\ldots,\ a_2x_3+b_2x_4+\ldots,\ \ldots,\ a_2x_m),$
. . . .
then $S_1S_2S_3\ldots \equiv (\alpha x_1+\beta x_2+\gamma x_3+\delta x_4+\ldots,\ \alpha x_2+\beta x_3+\gamma x_4+\ldots,$
$\alpha x_3+\beta x_4+\ldots,\ \ldots,\ \alpha x_m)$;
where $(a_1+b_1x+c_1x^2+\ldots)\times(a_2+b_2x+c_2x^2+\ldots)\times\ldots$
$\equiv(\alpha+\beta x+\gamma x^2+\ldots)$.

[Use induction.]

Ex. 9. Prove that any two of the substitutions S_1, S_2, S_3, ..., of Ex. 8 are permutable.

§ 3. Inverses and Powers.

The product AA is denoted by A^2, AAA by A^3, $AAAA$ by $A^4, \ldots$.

If the determinant of A does not vanish, we may solve the equations (i) of § 1 for $x_1, x_2, \ldots, x_m$. Then we obtain

$$x_t = \alpha_{t1}x_1'+\alpha_{t2}x_2'+\ \ldots\ +\alpha_{tm}x_m'\ (t=1,2,\ldots,m)$$

where $\alpha_{ij}=A_{ij}/|a|$; A_{ij} being the cofactor of a_{ij} in the expansion of the determinant $|a|$ of A.

The substitution

$$x_t'=\alpha_{t1}x_1+\alpha_{t2}x_2+\ \ldots\ +\alpha_{tm}x_m\ (t=1,2,\ldots,m)$$

is called the substitution *inverse* to A, and is denoted by A^{-1}.*

It follows from the definition that $AA^{-1}=A^{-1}A=E$.

The substitutions inverse to A^2, A^3, A^4, ... are denoted by A^{-2}, A^{-3}, A^{-4}, ...

Since

$$A^2A^{-2}=E=A\ A^{-1}=AEA^{-1}=A\,.\,A\ A^{-1}\,.\,A^{-1}$$
$$=A^2\,.\,A^{-1}A^{-1},\ (A^{-1})^2=A^{-2};$$

and similarly

$$(A^{-1})^3=A^{-3},\ (A^{-1})^4=A^{-4},\ \ldots.$$

The reader will now readily prove that (if $A^0=E$ and $A^1=A$) $A^mA^n=A^{m+n}$; where m and n are any positive or negative integers.

The inverse of AB is $B^{-1}A^{-1}$. In fact $AB\,.\,B^{-1}A^{-1}=AA^{-1}=E$. Similarly the inverse of ABC is $C^{-1}B^{-1}A^{-1}$; and so on.

The substitutions A, A^2, A^3, A^4, ... may be all distinct. If this is *not* the case, suppose $A^r=A^s$ $(r>s)$. Then $A^r\,.\,A^{-s}$

* This definition only holds good if the determinant of A is not zero.

$= A^s . A^{-s}$ or $A^{r-s} = E$; so that one of the series $A, A^2, A^3, A^4, \ldots$ is E. Let A^n be the *first* of this series which is E. Then n is called the *order* of A.

Ex. 1. Find the inverse of

$$x' = 4x+y+3z,\ y' = 4x-y+4z,\ z' = 3x+y+2z.$$

[Solving for x, y, z we get

$$x = -6x'+y'+7z',\ y = 4x'-y'-4z',\ z' = 7x'-y'-8z'.$$

Therefore the required inverse is

$$x' = -6x+y+7z,\ y' = 4x-y-4z,\ z' = 7x-y-8z.]$$

Ex. 2. Find the inverses of

$$(3x-2y,\ 4x-3y),\ (-2y-z,\ 2x+3y+2z,\ -x-y-z),$$
$$(2x+y+z+w,\ 2x+2z-w,\ -x+y-z+w,\ -y-w).$$

$$[(3x-2y,\ 4x-3y),\ (x+y+z,\ y+2z,\ -x-2y-4z),$$
$$(x+z+2w,\ y+2z+w,\ -x-2z-3w,\ -y-2z-2w).]$$

Ex. 3. Find the inverses of

$$(x_1,\ x_2-a_2x_1,\ x_3-a_3x_1, \ldots,\ x_m-a_mx_1),$$
$$(\alpha x_1+x_2,\ \alpha x_2+x_3,\ \ldots,\ \alpha x_{m-1}+x_m,\ \alpha x_m),$$
$$(x_2,\ x_3,\ \ldots,\ x_m,\ e_1x_1+e_2x_2+\ldots+e_mx_m),$$
$$(x_1,\ \alpha x_1+x_2,\ \alpha^2x_1+2\alpha x_2+x_3,\ \ldots,\ \alpha^{m-1}x_1+{}^{m-1}C_1\alpha^{m-2}x_2 + {}^{m-1}C_2\alpha^{m-3}x_3+\ldots+x_m).$$

$$[(x_1,\ x_2+a_2x_1,\ x_3+a_3x_1,\ \ldots,\ x_m+a_mx_1),$$
$$(\alpha^{-1}x_1-\alpha^{-2}x_2+\alpha^{-3}x_3-\ldots,\ \alpha^{-1}x_2-\alpha^{-2}x_3+\alpha^{-3}x_4-\ldots,\ \ldots,\ \alpha^{-1}x_m),$$
$$\left(\frac{1}{e_1}(-e_2x_1-e_3x_2-\ldots-e_mx_{m-1}+x_m),\ x_1,\ x_2,\ \ldots,\ x_{m-1}\right),$$
$$(x_1,\ -\alpha x_1+x_2,\ \alpha^2x_1-2\alpha x_2+x_3,\ -\alpha^3x_1+3\alpha^2x_2-3\alpha x_3+x_4,\ \ldots).]$$

Ex. 4. The inverse of a substitution with a matrix of the type

$$\begin{vmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha & \beta \\ 0 & \alpha & \beta & \gamma \\ \alpha & \beta & \gamma & \delta \end{vmatrix} \text{ has a matrix of the type } \begin{vmatrix} D & C & B & A \\ C & B & A & 0 \\ B & A & 0 & 0 \\ A & 0 & 0 & 0 \end{vmatrix};\ \text{and}$$

similarly for substitutions of any degree.

Ex. 5. Find the orders of

$$(3x-4y,\ 2x-3y),\ (8x-13y,\ 5x-8y),\ \left(\frac{1}{\sqrt{2}}(x-y),\ \frac{1}{\sqrt{2}}(x+y)\right),$$
$$(-y,\ -z,\ -x),\ (5x+6y,\ -4x-5y,\ 8x+8y-z),$$
$$(3x-3y+4z,\ 2x-3y+4z,\ -y+z),$$
$$(5x+2y+2z+2w,\ 2x+y+2w,\ -8x-4y-3z-4w,\ -6x-2y-2z-3w).$$

[2, 4, 8, 6, 2, 4, 2.]

Ex. 6. Show that

$$(-x_1-x_2-x_3-x_4-\ldots,\ x_2+{}^2C_1x_3+{}^3C_1x_4+\ldots,\ -x_3-{}^3C_2x_4-{}^4C_2x_5-\ldots,\ \ldots,\ (-1)^m x_m)$$

is of order 2.

Ex. 7. Show that

$$x_t' = x_t - \frac{2a_t}{a_1^2 + a_2^2 + \dots + a_m^2}(a_1x_1 + a_2x_2 + \dots + a_mx_m)$$

is of order 2.

What is the geometrical interpretation if $m = 3$?

[(x_1', x_2', x_3') and (x_1, x_2, x_3) are reflexions of each other in the plane $a_1x_1 + a_2x_2 + a_3x_3 = 0$.]

Ex. 8. If $ad - bc = 1$, the necessary and sufficient conditions that $(ax + by, cx + dy)$ should be of order 2 are $a^2 = 1$, $b = c = 0$. If $ad - bc = -1$, the condition is $a + d = 0$.

Ex. 9. If A is $(ax, bx + ay, cx + az)$,

$$A^n \text{ is } (a^nx, na^{n-1}bx + a^ny, na^{n-1}cx + a^nz).$$

[If this were true we should obtain A^{n+1} by substituting ax for x, $bx + ay$ for y, $cx + az$ for z in this value of A^n; i.e. A^{n+1} would be

$$(a^n(ax), na^{n-1}b(ax) + a^n(bx + ay), na^{n-1}c(ax) + a^n(cx + az))$$

or

$$(a^{n+1}x, (n+1)a^nbx + a^{n+1}y, (n+1)a^ncx + a^{n+1}z).$$

Now use induction. Examples 10 to 15 may be treated somewhat similarly.]

If A is of finite order, $b = c = 0$, and a is a root of unity.

Ex. 10. If A is $(ax, cx + dy)$, A^n is $\left(a^nx, \dfrac{a^n - d^n}{a - d}cx + d^ny\right)$.

What are the conditions that A is of finite order?

Ex. 11. If A is $(\cos\omega . x - \sin\omega . y, \sin\omega . x + \cos\omega . y)$, A^n is $(\cos n\omega . x - \sin n\omega . y, \sin n\omega . x + \cos n\omega . y)$.

What is the condition that A is of finite order?

Ex. 12. Prove that the n-th power of

$$(\alpha x_1 + x_2, \alpha x_2 + x_3, \dots, \alpha x_{m-1} + x_m, \alpha x_m)$$

is

$$((\alpha + \epsilon)^n x_1, (\alpha + \epsilon)^n x_2, \dots, \alpha^n x_m);$$

where ϵ is the operation changing x_r into x_{r+1}, and

$$x_{m+1} = x_{m+2} = \dots = 0.$$

The substitution is not of finite order if $m > 1$.

Ex. 13. Find the n-th power of

$$S \equiv (ax_1 + bx_2 + cx_3 + dx_4 + \dots, ax_2 + bx_3 + cx_4 + \dots, ax_3 + bx_4 + \dots, \dots, ax_m).$$

If S is of finite order, $b = c = d = \dots = 0$, and a is a root of unity.

[Use § 2, Ex. 8.]

Ex. 14. If $S \equiv (x_2, x_3, \dots, x_m, e_1x_1 + e_2x_2 + \dots + e_mx_m)$;

$S^2 = (x_3, x_4, x_5, \dots, x_m, a_my_m, a_my_{m-1} + a_{m+1}y_m)$,

$S^3 = (x_4, x_5, x_6, \dots, a_my_m, a_my_{m-1} + a_{m+1}y_m, a_my_{m-2} + a_{m+1}y_{m-1} + a_{m+2}y_m)$,

. . . .

$$S^m = (a_m y_m,\ a_m y_{m-1} + a_{m+1} y_m,\ \ldots,\ a_m y_1 + a_{m+1} y_2 + \ldots + a_{2m-1} y_m),$$
$$S^{m+1} = (a_m y_{m-1} + a_{m+1} y_m,\ a_m y_{m-2} + a_{m+1} y_{m-1} + a_{m+2} y_m,\ \ldots,$$
$$a_{m+1} y_1 + a_{m+2} y_2 + \ldots + a_{2m} y_m),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots;$$

where $y_1 = e_1 x_m,\ y_2 = e_1 x_{m-1} + e_2 x_m,\ \ldots,$
$$y_m = e_1 x_1 + e_2 x_2 + \ldots + e_m x_m,$$
and $a_1 \lambda + a_2 \lambda^2 + a_3 \lambda^3 + \ldots + a_m \lambda^m + a_{m+1} \lambda^{m+1} + \ldots$
$$\equiv \frac{e_1 \lambda^m}{(-\lambda^m + e_1 + e_2 \lambda + \ldots + e_m \lambda^{m-1})}.$$

Ex. 15. If $s_{tt} = 1 + k a_{tt},\ s_{ij} = k a_{ij}\ (i \neq j)$; and $S^{n+1} = P$, then
$$p_{tt} = (k a_{tt} + {}^nC_1 k^2 b_{tt} + {}^nC_2 k^3 c_{tt} + \ldots) + (1 + k a_{tt})^n,$$
$$p_{ij} = (k a_{ij} + {}^nC_1 k^2 b_{ij} + {}^nC_2 k^3 c_{ij} + \ldots);$$
where $B = A^2,\ C = A^3,\ \ldots.$

If k and the a's are integers and S is of finite order, $S = E$ unless $k = \pm 2$. If $k = \pm 2$, $S = E$ or $S^2 = E$. (Minkowsky's Theorem).

Ex. 16. The determinant of a substitution of order n is an n-th root of unity.

[The determinant of A^n is the n-th power of the determinant of A.]

Ex. 17. If A is of order n, $A^r = E$, if and only if $r = kn$; where k is a positive or negative integer.

Ex. 18. If A is of order n, A^r is of order n/d; where d is the H. C. F. of n and r.

Ex. 19. If A and B are permutable, so are A^r and B^s.

Ex. 20. If $A, B, C, \ldots$ are permutable, the order of $ABC\ldots$ is the least common multiple of the orders of $A, B, C, \ldots.$

§ 4. Transposed and Conjugate Substitutions.

Let A be the substitution with matrix

$$\begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mm} \end{vmatrix}.$$

The substitution with matrix

$$\begin{vmatrix} a_{11} & a_{21} & \cdot & \cdot & \cdot & a_{m1} \\ a_{12} & a_{22} & \cdot & \cdot & \cdot & a_{m2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{1m} & a_{2m} & \cdot & \cdot & \cdot & a_{mm} \end{vmatrix}$$

is called the substitution *transposed* to A, and is denoted by A'.

The substitution with matrix

$$\begin{vmatrix} \bar{a}_{11} & \bar{a}_{12} & . & . & . & \bar{a}_{1m} \\ \bar{a}_{21} & \bar{a}_{22} & . & . & . & \bar{a}_{2m} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \bar{a}_{m1} & \bar{a}_{m2} & . & . & . & \bar{a}_{mm} \end{vmatrix}$$

is called the substitution *conjugate* to A, and is denoted by $\bar{A}$. In this $\bar{a}_{ij}$ denotes the complex quantity conjugate to a_{ij}.* We shall adopt this notation throughout.

The substitution conjugate to A' is denoted by $\bar{A}'$.

It follows at once from equation (i) of § 2 that with the notation there used $B'A'$ is

$$x_t' = p_{1t}x_1 + p_{2t}x_2 + \ldots + p_{mt}x_m;$$

or that $B'A'$ is the substitution transposed to AB.

It can now be at once proved that the substitution transposed to ABC is $C'B'A'$; to $ABCD$ is $D'C'B'A'$, &c.

Similarly the substitution conjugate to AB is $\bar{A}\bar{B}$, to ABC is $\bar{A}\bar{B}\bar{C}$, &c.

It follows that A'^2, A'^3, ... are the substitutions transposed to A^2, A^3, ...; and $\bar{A}^2$, $\bar{A}^3$, ... are the substitutions conjugate to A^2, A^3, ...

Again, the substitution transposed to $E = AA^{-1}$ is $(A^{-1})'A'$. But E is transposed to itself, and hence $(A^{-1})'A' = E$ or $(A')^{-1} = (A^{-1})'$.

Therefore the substitution transposed to the inverse of A is the inverse of the substitution transposed to A; and similarly, the conjugate of the inverse is the inverse of the conjugate.

Ex. 1. The determinant of A is the same as that of A', and is conjugate to that of $\bar{A}$ and $\bar{A}'$.

Ex. 2. If A and B are permutable, so are A' and B', and so are $\bar{A}$ and $\bar{B}$.

§ 5. Transformation of Substitutions.

If $R = S^{-1}AS$, R is called the *transform of A by S.* The importance of this transform lies in the theorem :—

The transform of A by S is found by expressing A in terms of new variables defined by S.

* If a_{ij} is real, $\bar{a}_{ij} = a_{ij}$.

In other words, if in the equations

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \; (t = 1, 2, \ldots, m)$$

defining A we put

$$\left.\begin{array}{l} s_{t1}x_1' + s_{t2}x_2' + \ldots + s_{tm}x_m' \text{ equal to } y_t' \\ s_{t1}x_1 \;\, + s_{t2}x_2 \;\, + \ldots + s_{tm}x_m \;\, \text{ equal to } y_t \end{array}\right\} (t = 1, 2, \ldots, m)$$

and then solve for $y_1', y_2', \ldots, y_m'$ in terms of $y_1, y_2, \ldots, y_m$, we get

$$y_t' = r_{t1}y_1 + r_{t2}y_2 + \ldots + r_{tm}y_m \; (t = 1, 2, \ldots, m).$$

In fact, we have by § 2 if $S^{-1}A = D$, i.e. $A = SD$,

$$x_t' = d_{t1}y_1 + d_{t2}y_2 + \ldots + d_{tm}y_m \; (t = 1, 2, \ldots, m).$$

Hence

$$y_t' = s_{t1}x_1' + s_{t2}x_2' + \ldots + s_{tm}x_m' = r_{t1}y_1 + r_{t2}y_2 + \ldots + r_{tm}y_m;$$

since $R = DS$ gives

$$r_{ij} = s_{i1}d_{1j} + s_{i2}d_{2j} + \ldots + s_{im}d_{mj}.$$

The result of transforming A by several substitutions in succession is equivalent to transforming A by a single substitution.

For example, the result of transforming A in succession by S_1, S_2, S_3 is $S_3^{-1}\{S_2^{-1}(S_1^{-1}AS_1)S_2\}S_3$, which is

$$(S_1S_2S_3)^{-1}A(S_1S_2S_3).$$

The transform of a product of any substitutions by a given substitution S is the product of their transforms by S.

For example,

$$\begin{aligned} S^{-1}ABS &= S^{-1}AEBS = S^{-1}AS \,.\, S^{-1}BS; \\ S^{-1}ABCS &= S^{-1}AS \,.\, S^{-1}BS \,.\, S^{-1}CS; \text{ and so on.} \end{aligned}$$

In particular

$$\begin{aligned} &S^{-1}A^2S = S^{-1}AS \,.\, S^{-1}AS = (S^{-1}AS)^2; \\ &S^{-1}A^3S = (S^{-1}AS)^3; \text{ and so on.} \end{aligned}$$

The transform of the inverse of A by S is the inverse of the transform of A by S.

For since

$$S^{-1}AS \,.\, S^{-1}A^{-1}S = S^{-1}AA^{-1}S = S^{-1}S = E,$$

$S^{-1}A^{-1}S$ is the inverse of $S^{-1}AS$.

It follows that $S^{-1}A^{-k}S = (S^{-1}A^{-1}S)^k$.

If a substitution A is of finite order n, so is any transform of A.

For $(S^{-1}AS)^n$ or $S^{-1}A^nS = E$ if and only if $A^n = E$.

Ex. 1. Transform $A \equiv (-7x-4y-9z,\ -2x-z,\ 6x+3y+7z)$ by $S \equiv (3x+2y+4z,\ -x-z,\ x+y+2z)$.

Deduce the order of A.

[$S^{-1} \equiv (x-2z,\ x+2y-z,\ -x-y+2z)$. Hence by § 2

$$AS \equiv (-x-z,\ x+y+2z,\ 3x+2y+4z),$$

and thence
$$S^{-1}.AS \equiv (y,\ z,\ x).$$

The order of this is evidently 3, and hence A is of order 3.]

Ex. 2. Show that the transform

of $(2y, x)$ by $(2x-y,\ -3x+2y)$ is $(10x+7y,\ -14x-10y)$,

of $(x,\ 2y,\ -z)$ by $(x-y,\ 2x-y+z,\ 2x-2y+z)$

is $(x-y+z,\ 4x-z,\ 4x-2y+z)$,

of $(-x+y,\ -y,\ -z+w,\ -w)$

by $(x+z+2w,\ y+2z+w,\ -x-2z-3w,\ -y-2z-2w)$

is $(x-y+2z-2w,\ -3y-2w,\ -2x+2y-3z+3w,\ 2y+w)$.

Ex. 3. The transform of $(\omega_1 x_1,\ \epsilon_2 x_1+\omega_2 x_2,\ \ldots,\ \epsilon_m x_1+\omega_m x_m)$

by $\left(x_1,\ \frac{\epsilon_2}{\omega_2-\omega_1} x_1+x_2,\ \ldots,\ \frac{\epsilon_m}{\omega_m-\omega_1} x_1+x_m\right)$

is $(\omega_1 x_1,\ \omega_2 x_2,\ \ldots,\ \omega_m x_m)$.

Ex. 4. (i) Express $A \equiv (x_2,\ x_3,\ \ldots,\ x_m,\ e_1 x_1+e_2 x_2+\ldots+e_m x_m)$ in terms of new variables $y_1, y_2, \ldots, y_m$, where

$y_1 = e_1 x_m,\ y_2 = e_2 x_m+e_1 x_{m-1},\ \ldots,\ y_m = e_m x_m+e_{m-1}x_{m-1}+\ldots+e_1 x_1$.

[$y_1' = e_1 y_m,\ y_2' = e_2 y_m+y_1,\ \ldots,\ y_m' = e_m y_m+y_{m-1}$.]

(ii) If $B \equiv (e_1 x_m,\ e_2 x_m+e_1 x_{m-1},\ \ldots,\ e_m x_m+e_{m-1}x_{m-1}+\ldots+e_1 x_1)$, show that $B^{-1}AB = A'$.

[By § 5, $B^{-1}AB$ is $(e_1 x_m,\ e_2 x_m+x_1,\ \ldots,\ e_m x_m+x_{m-1})$, which is A'.]

Ex. 5. If $A \equiv (x_1+x_2,\ x_2+x_3,\ \ldots,\ x_{m-1}+x_m,\ x_m)$

and $B \equiv (-x_1+x_2-x_3+x_4-\ldots,\ x_2-{}^2C_1 x_3+{}^3C_1 x_4-,$

$-x_3+{}^3C_2 x_4-{}^4C_2 x_5+\ldots,\ \ldots,\ (-1)^m x_m)$,

show that $B^{-1}AB = A^{-1}$.

Ex. 6. AB and BA have the same order.

[$B^{-1}.BA.B = AB$.]

Ex. 7. If the transforms of C by A and B are the same, BA^{-1} and AB^{-1} are permutable with C.

Ex. 8. If $B^{-1}AB = A^k$, $B^{-\beta}A^{\alpha}B^{\beta} = A^{\alpha k^\beta}$.

[Use induction].

§ 6. Poles and Characteristic-equations.

Quantities $X_1, X_2, \ldots, X_m$ not all zero, such that

$$\lambda X_t = a_{t1}X_1+a_{t2}X_2+\ldots+a_{tm}X_m \quad (t = 1, 2, \ldots, m) \ldots\ldots \text{(i)}$$

are said to define a *pole* of A. They are values of $x_1, x_2, \ldots, x_m$ making
$$x_1' : x_2' : \ldots : x_m' = x_1 : x_2 : \ldots : x_m$$

in the equations

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \dots + a_{tm}x_m \quad (t = 1, 2, \dots, m),$$

which define the substitution A.

We are only concerned with the *ratios* of $X_1, X_2, \dots, X_m$; i.e. two poles $(X_1, X_2, \dots, X_m)$ and $(Z_1, Z_2, \dots, Z_m)$ are not considered distinct if

$$X_1 : X_2 : \dots : X_m = Z_1 : Z_2 : \dots : Z_m.$$

Eliminating $X_1, X_2, \dots, X_m$ from the m equations (i) we get

$$\theta(\lambda) \equiv \begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22}-\lambda & \dots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mm}-\lambda \end{vmatrix} = 0.$$

We call $\theta(\lambda)$ the *characteristic-determinant* of A, and $\theta(\lambda) = 0$ the *characteristic-equation* of A. Any root of the characteristic-equation is called a *characteristic-root* of A.

A substitution has the same characteristic-determinant as any substitution into which it can be transformed.

Suppose $R = S^{-1}AS$; it is required to prove that

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22}-\lambda & \dots & a_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mm}-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} r_{11}-\lambda & r_{12} & \dots & r_{1m} \\ r_{21} & r_{22}-\lambda & \dots & r_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r_{m1} & r_{m2} & \dots & r_{mm}-\lambda \end{vmatrix}.$$

In fact, since $AS = SR$ we have by § 2

$$s_{i1}a_{1j} + s_{i2}a_{2j} + \dots + s_{im}a_{mj} = r_{i1}s_{1j} + r_{i2}s_{2j} + \dots + r_{im}s_{mj};$$

and therefore by the ordinary rule for the multiplication of determinants

$$\begin{vmatrix} s_{11} & s_{12} & \cdot & \cdot & \cdot & s_{1m} \\ s_{21} & s_{22} & \cdot & \cdot & \cdot & s_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{m1} & s_{m2} & \cdot & \cdot & \cdot & s_{mm} \end{vmatrix} \times \begin{vmatrix} a_{11}-\lambda & a_{21} & \cdot & \cdot & \cdot & a_{m1} \\ a_{12} & a_{22}-\lambda & \cdot & \cdot & \cdot & a_{m2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{1m} & a_{2m} & \cdot & \cdot & \cdot & a_{mm}-\lambda \end{vmatrix}$$

$$= \begin{vmatrix} r_{11}-\lambda & r_{12} & \cdot & \cdot & \cdot & r_{1m} \\ r_{21} & r_{22}-\lambda & \cdot & \cdot & \cdot & r_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r_{m1} & r_{m2} & \cdot & \cdot & \cdot & r_{mm}-\lambda \end{vmatrix}$$

$$\times \begin{vmatrix} s_{11} & s_{21} & \cdot & \cdot & \cdot & s_{m1} \\ s_{12} & s_{22} & \cdot & \cdot & \cdot & s_{m2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{1m} & s_{2m} & \cdot & \cdot & \cdot & s_{mm} \end{vmatrix}.$$

To each distinct characteristic-root of A corresponds at least one pole. We now show that:—

If we apply the substitution S to a pole of A, we get a pole of $S^{-1}AS$ corresponding to the same common characteristic-root of A and $S^{-1}AS$.

In other words, if $(X_1, X_2, \ldots, X_m)$ is a pole of A, then $(Y_1, Y_2, \ldots, Y_m)$ is a pole of $R = S^{-1}AS$ corresponding to the same characteristic root, where

$$\left.\begin{aligned} Y_1 &= s_{11}X_1 + s_{12}X_2 + \ldots + s_{1m}X_m \\ Y_2 &= s_{21}X_1 + s_{22}X_2 + \ldots + s_{2m}X_m \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ Y_m &= s_{m1}X_1 + s_{m2}X_2 + \ldots + s_{mm}X_m \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

For by § 5 if

$$\left.\begin{aligned} X_t' &= a_{t1}X_1 + a_{t2}X_2 + \ldots + a_{tm}X_m \\ Y_t' &= s_{t1}X_1' + s_{t2}X_2' + \ldots + s_{tm}X_m' \\ Y_t &= s_{t1}X_1 + s_{t2}X_2 + \ldots + s_{tm}X_m \end{aligned}\right\} (t = 1, 2, \ldots, m),$$

then $$Y_t' = r_{t1}Y_1 + r_{t2}Y_2 + \ldots + r_{tm}Y_m.$$

Now since $(X_1, X_2, \ldots, X_m)$ is a pole of A, $X_t' = \lambda X_t$.

Therefore $Y_t' = \lambda Y_t$, and $(Y_1, Y_2, \ldots, Y_m)$ is a pole of R.

We verify at once that A has $(1, 0, 0, \ldots, 0)$ as a pole if and only if $a_{21} = a_{31} = \ldots = a_{m1} = 0$.

It is evident that we can choose S so that $S^{-1}AS$ has $(1, 0, 0, \ldots, 0)$ as a pole. We have only to choose S to satisfy the conditions

$$s_{t1}X_1 + s_{t2}X_2 + \ldots + s_{tm}X_m = 0 \ (t = 2, 3, \ldots, m),$$

where $(X_1, X_2, \ldots, X_m)$ is a pole of A.

In fact, if $X_k \neq 0$ we may take S, for instance, as

$$\begin{aligned} &x_1' = -X_k x_k, \quad x_2' = -X_2 x_k + X_k x_2, \quad \ldots, \\ &\quad x'_{k-1} = -X_{k-1}x_k + X_k x_{k-1}, \quad x_k' = -X_1 x_k + X_k x_1, \\ &\qquad x'_{k+1} = -X_{k+1}x_k + X_k x_{k+1}, \quad \ldots, \quad x_m' = -X_m x_k + X_k x_m, \end{aligned}$$

which has the determinant $\pm X_k^m$.*

We have now transformed A into $S^{-1}AS = R$, where R is

$$\left.\begin{aligned} x_1' &= r_{11}x_1 + r_{12}x_2 + r_{13}x_3 + \ldots + r_{1m}x_m \\ x_t' &= \qquad\quad r_{t2}x_2 + r_{t3}x_3 + \ldots + r_{tm}x_m \ (t = 2, 3, \ldots, m) \end{aligned}\right\}$$

If we now transform the substitution

$$x_t' = r_{t2}x_2 + r_{t3}x_3 + \ldots + r_{tm}x_m \ (t = 2, 3, \ldots, m)$$

similarly, and repeat the process as often as required, we transform A into a substitution of the type

$$\left.\begin{array}{ll} x_1' = & k_{11}x_1 + k_{12}x_2 + k_{13}x_3 + \ldots + k_{1m}x_m \\ x_2' = & \qquad\quad\ k_{22}x_2 + k_{23}x_3 + \ldots + k_{2m}x_m \\ x_3' = & \qquad\qquad\qquad\ \ k_{33}x_3 + \ldots + k_{3m}x_m \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_m' = & \qquad\qquad\qquad\qquad\qquad\quad k_{mm}x_m \end{array}\right\},$$

which is sometimes called a *normal* substitution.

An alternative method of transforming A into a normal substitution is the following:—

Let $(Z_1, Z_2, \ldots, Z_m)$ be a pole of A' corresponding to a characteristic-root α of A'.

If $\xi_m \equiv Z_1x_1 + Z_2x_2 + \ldots + Z_m x_m$, $\xi_m' = \alpha\xi_m$ by the definition of a pole.

Hence the transform of A by

$$\begin{aligned} &x_1' = x_1, \quad x_2' = x_2, \quad \ldots, \quad x'_{m-1} = x_m, \\ &\qquad\qquad\qquad\qquad x_m' = Z_1x_1 + Z_2x_2 + \ldots + Z_m x_m \end{aligned}$$

is of the type †

$$\left.\begin{array}{ll} x_1' &= e_{11}x_1 + e_{12}x_2 + \ldots + e_{1\,m-1}x_{m-1} + e_{1m}x_m \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x'_{m-1} &= e_{m-1\,1}x_1 + e_{m-1\,2}x_2 + \ldots + e_{m-1\,m-1}x_{m-1} + e_{m-1\,m}x_m \\ x_m' &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \alpha x_m \end{array}\right\}.$$

* If $X_1 \neq 0$, we take S as $x_1' = -X_1x_1$, $x_2' = -X_2x_1 + X_1x_2$, $\ldots$, $x_m' = -X_m x_1 + X_1 x_m$, which is of order 2, if $X_1^2 = 1$.

† We assume $Z_m \neq 0$; but there is no loss of generality.

We now transform

$$x_t' = e_{t1}x_1 + e_{t2}x_2 + \dots + e_{t\,m-1}x_{m-1}\ (t = 1, 2, \dots, m-1)$$

in a similar manner; and repeat the process till A is transformed into *normal* type.

Ex. 1. Find the poles of $(2y+5z,\ -z,\ -x+4z)$.

[If (X, Y, Z) is a pole corresponding to the characteristic-root λ,

$$-\lambda X+2Y+5Z = -\lambda Y-Z = -X+(4-\lambda)Z = 0.$$

These give on elimination of X, Y, Z the characteristic-equation

$$\begin{vmatrix} -\lambda & 2 & 5 \\ 0 & -\lambda & -1 \\ -1 & 0 & 4-\lambda \end{vmatrix} = 0, \text{ or } (\lambda-1)^2(\lambda-2) = 0.$$

Putting $\lambda = 2$ we get

$$-2X+2Y+5Z = -2Y-Z = -X+2Z = 0,$$

which are satisfied only by $X:Y:Z = 4:-1:2$.

Putting $\lambda = 1$ we get

$$-X+2Y+5Z = -Y-Z = -X+3Z = 0,$$

which are satisfied only by $X:Y:Z = 3:-1:1$.

Hence there are two poles, $(4, -1, 2)$ and $(3, -1, 1)$.]

Ex. 2. Find the poles of

$$(15x-4y-16z,\ 16x-5y-16z,\ 12x-3y-13z).$$

[The poles are given by

$$(15-\lambda)X-4Y-16Z = 16x-(5+\lambda)Y-16Z$$
$$= 12X-3Y-(13+\lambda)Z = 0.$$

Hence the characteristic-equation is $(\lambda+1)^3 = 0$; and when we put $\lambda = -1$ these three equations all reduce to $4X-Y-4Z = 0$. Therefore there is a singly infinite number of poles $(X, 4X-4, 1)$; where X may have any value.]

Ex. 3. Find the poles of $R \equiv (x-y+z,\ 4x-z,\ 4x-2y+z)$.

[The characteristic-roots $1, 2, -1$ give respectively the poles $(1, 2, 2)$, $(1, 1, 2)$, $(0, 1, 1)$.

To verify, notice that $R = S^{-1}AS$, where $A \equiv (x, 2y, -z)$, $S \equiv (x-y,\ 2x-y+z,\ 2x-2y+z)$. Applying the substitution S to the three poles of A, namely $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, we get the poles just obtained.]

Ex. 4. Find the poles of

$$(x+2y,\ x),\ (3x+y-2z,\ 2x+3y-3z,\ 4x+2y-3z),$$
$$(x-y+2z-2w,\ -3y-2w,\ -2x+2y-3z+3w,\ 2y+w).$$

[(i) $\lambda = 2$ gives $(2, 1)$; $\lambda = -1$ gives $(1, -1)$.

(ii) $\lambda = 1$ gives $(1, 2, 2)$.

(iii) $\lambda = -1$ gives $(X, -2, 1-X, 2)$.]

Ex. 5. Find the poles of

$$(\cos\omega\,.\,x-\sin\omega\,.\,y,\ \sin\omega\,.\,x+\cos\omega\,.\,y),$$
$$(\cos\omega\,.\,x-\sin\omega\,.\,y,\ -\sin\omega\,.\,x-\cos\omega\,.\,y),$$
$$(\alpha x_1+x_2,\ \alpha x_2+x_3,\ \ldots,\ \alpha x_{m-1}+x_m,\ \alpha x_m),$$
$$(x_2,\ x_3,\ \ldots,\ x_m,\ e_1x_1+e_2x_2+\ldots+e_mx_m).$$

[(i) $\lambda=\cos\omega+i\sin\omega$ gives $(i,\ 1)$; $\lambda=\cos\omega-i\sin\omega$ gives $(-i,\ 1)$.

(ii) $\lambda=1$ gives $(\cos\frac{1}{2}\omega,\ -\sin\frac{1}{2}\omega)$; $\lambda=-1$ gives $(\sin\frac{1}{2}\omega,\ \cos\frac{1}{2}\omega)$.

(iii) $\lambda=\alpha$ gives $(1,\ 0,\ 0,\ \ldots,\ 0)$.

(iv) $(\lambda,\ \lambda^2,\lambda^3,\ \ldots,\ \lambda^m)$ where λ is any root of the characteristic-equation $\lambda^m=e_1+e_2\lambda+\ldots+e_m\lambda^{m-1}$.]

Ex. 6. Find the poles of

$$x_t'=x_t+k_t(c_1x_1+c_2x_2+\ldots+c_mx_m),\quad (t=1,\ 2,\ \ldots,\ m).$$

[The characteristic-equation is

$$(\lambda-1)^{m-1}(\lambda-1-k_1c_1-k_2c_2-\ldots-k_mc_m)=0.$$

The poles are

$$(c_mX_1,\ c_mX_2,\ \ldots,\ c_mX_{m-1},\ -c_1X_1-c_2X_2-\ldots-c_{m-1}X_{m-1}),$$
$$(k_1,\ k_2,\ \ldots,\ k_m).]$$

Ex. 7. Find the poles of

$$x_t'=(a_1^2+a_2^2+\ldots+a_m^2)x_t-2a_t(a_1x_1+a_2x_2+\ldots+a_mx_m).$$

[The characteristic-equation is

$$(\lambda-a_1^2-a_2^2-\ldots-a_m^2)^{m-1}(\lambda+a_1^2+a_2^2+\ldots+a_m^2)=0.$$

The poles are $(X_1,\ X_2,\ \ldots,\ X_m)$ where $a_1X_1+a_2X_2+\ldots+a_mX_m=0$, and $(a_1,\ a_2,\ \ldots,\ a_m)$.]

Ex. 8. Find the poles of $(x_m,\ x_{m-1},\ \ldots,\ x_2,\ x_1)$.

[$\lambda=\pm1$.]

Ex. 9. Find the poles of a cyclant substitution (see § 2, Ex. 7).

[(i) Type I. If $\omega^m=1$, $\alpha_1+\alpha_2\omega+\alpha_3\omega^2+\ldots+\alpha_m\omega^{m-1}$ is a characteristic-root, and $(\omega,\ \omega^2,\ \ldots,\ \omega^m)$ the corresponding pole.

(ii) Type II. ab is a characteristic-root and

$$(a+b,\ \omega^{-1}a+\omega b,\ \omega^{-2}a+\omega^2b,\ \ldots,\ \omega^{-m+1}a+\omega^{m-1}b)$$

the pole; where a denotes $(\alpha_1+\alpha_2\omega+\alpha_3\omega^2+\ldots+\alpha_m\omega^{m-1})^{\frac{1}{2}}$ and b denotes $\pm(\alpha_1+\alpha_2\omega^{-1}+\alpha_3\omega^{-2}+\ldots+\alpha_m\omega^{-m+1})^{\frac{1}{2}}$.]

Ex. 10. Find the poles of

$$(-2x_1+x_2,\ x_1-2x_2+x_3,\ \ldots,\ x_{m-2}-2x_{m-1}+x_m,\ x_{m-1}-2x_m).$$

[$\lambda=-4\cos^2\frac{1}{2}\alpha$ gives $(-\sin m\alpha,\ +\sin\overline{m-1}\alpha,$
$-\sin\overline{m-2}\alpha,\ \ldots,\ (-1)^m\sin\alpha)$;

where $(m+1)\alpha=r\pi,\ (r=1,\ 2,\ \ldots,\ m)$.]

Ex. 11. The product of the characteristic-roots of A is the determinant of A, and their sum is $a_{11}+a_{22}+\ldots+a_{mm}$.

Ex. 12. If $(X_1, X_2, \ldots, X_m)$ is a pole of A corresponding to a characteristic-root α, and also a pole of B corresponding to a characteristic-root β, it is a pole of AB corresponding to the characteristic-root $\alpha\beta$.

[Transform the substitutions so that the given pole becomes $(1, 0, 0, \ldots, 0)$.]

Ex. 13. If $CB = A$, the characteristic-equation of C is

$$\begin{vmatrix} a_{11}-\lambda b_{11} & a_{12}-\lambda b_{12} & \cdot & \cdot & \cdot & a_{1m}-\lambda b_{1m} \\ a_{21}-\lambda b_{21} & a_{22}-\lambda b_{22} & \cdot & \cdot & \cdot & a_{2m}-\lambda b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1}-\lambda b_{m1} & a_{m2}-\lambda b_{m2} & \cdot & \cdot & \cdot & a_{mm}-\lambda b_{mm} \end{vmatrix} = 0.$$

[If $x_t' = c_{t1}x_1 + c_{t2}x_2 + \ldots + c_{tm}x_m$, then

$$b_{t1}x_1' + b_{t2}x_2' + \ldots + b_{tm}x_m' = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m.$$

Now put λX_t for x_t', X_t for x_t in these equations and eliminate $X_1, X_2, \ldots, X_m$.]

Ex. 14. The characteristic-roots of a *normal* substitution are the elements of the leading diagonal in its matrix.

Ex. 15. Transform into a normal substitution

$$A \equiv (x-y+2z-2w,\ -3y-2w,\ -2x+2y-3z+3w,\ 2y+w).$$

[The characteristic-equation is $(\lambda+1)^4 = 0$.

A pole of the transposed substitution is $(0, 1, 0, 1)$.

Transform therefore by $R \equiv (x, y, z, y+w)$, and we obtain

$$(x+y+2z-2w,\ -y-2w,\ -2x-y-3z+3w,\ -w).$$

A pole of the substitution transposed to

$$(x+y+2z,\ -y,\ -2x-y-3z) \text{ is } (1, 0, 1).$$

Transform therefore by $S \equiv (x, y, x+z, w)$, and we get

$$(-x+y+2z-2w,\ -y-2w,\ -z+w,\ -w),$$

which is normal.

Since $S^{-1}(R^{-1}AR)S$ is normal, A is transformed into normal form by $RS \equiv (x, y, x+z, y+w)$.]

Ex. 16. Transform into normal substitutions the other substitutions in Ex. 4.

Ex. 17. Find the condition that two given substitutions should have a common characteristic-root.

[If the matrices of the substitutions are

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ and } \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}, \text{ the condition is}$$

$$\begin{vmatrix} a_{11}-b_{11} & -b_{12} & a_{12} & 0 & a_{13} & 0 \\ -b_{21} & a_{11}-b_{22} & 0 & a_{12} & 0 & a_{13} \\ a_{21} & 0 & a_{22}-b_{11} & -b_{12} & a_{23} & 0 \\ 0 & a_{21} & -b_{21} & a_{22}-b_{22} & 0 & a_{23} \\ a_{31} & 0 & a_{32} & 0 & a_{33}-b_{11} & -b_{12} \\ 0 & a_{31} & 0 & a_{32} & -b_{21} & a_{33}-b_{22} \end{vmatrix} = 0,$$

and so in general.]

Ex. 18. Find the condition that two equations of the type given in § 13 should have a root in common.

§ 7. Multiplications, &c.

The substitution

$$x_1' = a_1x_1,\ x_2' = a_2x_2,\ \ldots,\ x_m' = a_mx_m$$

is called a *multiplication*.

Any two multiplications are evidently permutable, and their product is a multiplication.

The substitution

$$x_1' = ax_1,\ x_2' = ax_2,\ \ldots,\ x_m' = ax_m$$

is called a *similarity-substitution*.

It is evidently permutable with any substitution whatever. The product of two similarity-substitutions is a similarity-substitution.

The substitution

$$x_1' = a_1x_\alpha,\ x_2' = a_2x_\beta,\ \ldots\ x_m' = a_mx_\mu,$$

where $\alpha, \beta, \ldots, \mu$ are the symbols $1, 2, \ldots, m$ in some order or other, is called a *monomial* substitution.

If $a_1 = a_2 = \ldots = a_m = 1$, the monomial substitution is called a *permutation*.

The substitution of the type

$$\left.\begin{array}{lll} x_1' & = a_{11}x_1 & + \ldots + a_{1e}x_e \\ \vdots & \vdots & \vdots \\ x_e' & = a_{e1}x_1 & + \ldots + a_{ee}x_e \\ x'_{e+1} & = b_{11}x_{e+1} & + \ldots + b_{1f}x_{e+f} \\ \vdots & \vdots & \vdots \\ x'_{e+f} & = b_{f1}x_{e+1} & + \ldots + b_{ff}x_{e+f} \\ x'_{e+f+1} & = c_{11}x_{e+f+1} & + \ldots + c_{1g}x_{e+f+g} \\ \vdots & \vdots & \vdots \\ x'_{e+f+g} & = c_{g1}x_{e+f+1} & + \ldots + c_{gg}x_{e+f+g} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{array}\right\}$$

will be called the *direct product of the constituent substitutions*

$$\left.\begin{aligned} x_1' &= a_{11}x_1 + \dots + a_{1e}x_e \\ &\vdots \\ x_e' &= a_{e1}x_1 + \dots + a_{ee}x_e \end{aligned}\right\}, \quad \left.\begin{aligned} x'_{e+1} &= b_{11}x_{e+1} + \dots + b_{1f}x_{e+f} \\ &\vdots \\ x'_{e+f} &= b_{f1}x_{e+1} + \dots + b_{ff}x_{e+f} \end{aligned}\right\}, \dots.$$

It will be noticed that no two of the constituents have a variable in common.

If P is the direct product of $A, B, C, \dots$, then P^k is evidently the direct product of $A^k, B^k, C^k, \dots$, where k is a positive or negative integer.

The characteristic-equation and the poles of P are at once written down when those of $A, B, C, \dots$ are known. In fact, if $A, B, C, \dots$ have characteristic-determinants $\theta_1(\lambda)$, $\theta_2(\lambda)$, $\theta_3(\lambda), \dots$, the characteristic-determinant of P is

$$\theta_1(\lambda) \times \theta_2(\lambda) \times \theta_3(\lambda) \times \dots.$$

Again, if $A, B, C, \dots$ have a common characteristic-root λ_1, and the corresponding poles of $A, B, C, \dots$ are $(X_1, X_2, \dots, X_e)$, $(Y_1, Y_2, \dots, Y_f)$, $(Z_1, Z_2, \dots, Z_g), \dots$, then the poles of P corresponding to λ_1 are

$$(\alpha X_1, \alpha X_2, \dots, \alpha X_e, \beta Y_1, \beta Y_2, \dots, \beta Y_f, \gamma Z_1, \gamma Z_2, \dots, \gamma Z_g, \dots),$$

any values whatever of the ratios $\alpha : \beta : \gamma : \dots$ being taken.

If λ_1 is a characteristic-root of A and B, but not of $C, D, \dots$, the corresponding poles of P are

$$(\alpha X_1, \alpha X_2, \dots, \alpha X_e, \beta Y_1, \beta Y_2, \dots, \beta Y_f, 0, 0, \dots, 0, 0, \dots);$$

and so on in general.

Ex. 1. The coefficients of a multiplication of order n are n-th roots of unity.

Ex. 2. A substitution with

$$(1, 0, 0, \dots, 0),\ (0, 1, 0, \dots, 0),\ \dots,\ (0, 0, 0, \dots, 1)$$

as poles is a multiplication; and conversely.

Ex. 3. If two coefficients of a multiplication are equal, the substitution has an infinite number of poles.

Ex. 4. The coefficients of a multiplication are its characteristic-roots.

Ex. 5. If any substitution is multiplied by a similarity, its poles are unaltered and its characteristic-roots are all multiplied by the same quantity.

Ex. 6. The product of the coefficients of a monomial substitution of order n is an n-th root of unity.

Ex. 7. A permutation is always of finite order.

Ex. 8. The order of a direct product is the least common multiple of the orders of the constituent substitutions.

§ 8.

A substitution C of the type

$$x_1'=\alpha x_1+x_2,\ x_2'=\alpha x_2+x_3,\ \ldots,\ x'_{m-1}=\alpha x_{m-1}+x_m,\ x_m'=\alpha x_m$$

is of great importance in the theory of substitutions; for, as we shall prove in § 9, any substitution whatever can be transformed into the direct product of substitutions of this type.

It has the characteristic-determinant $(\alpha-\lambda)^m$.

If $(X_1, X_2, \ldots, X_m)$ is a pole,

$$\alpha X_1=\alpha X_1+X_2,\ \ldots,\ \alpha X_{m-1}=\alpha X_{m-1}+X_m,\ \alpha X_m=\alpha X_m;$$

which give $\qquad X_2=X_3=\ldots=X_m=0.$

Hence C has only one distinct pole $(1, 0, 0, \ldots, 0)$.

The k-th power of C is

$$\left.\begin{array}{ll}
x_1' = & \alpha^k x_1+{}^kC_1\alpha^{k-1}x_2+{}^kC_2\alpha^{k-2}x_3+{}^kC_3\alpha^{k-3}x_4+\ldots \\
 & \qquad\qquad\qquad\ldots+{}^kC_{m-1}\alpha^{k-m+1}x_m \\
x_2' = & \qquad\qquad \alpha^k x_2+{}^kC_1\alpha^{k-1}x_3+{}^kC_2\alpha^{k-2}x_4+\ldots \\
 & \qquad\qquad\qquad\ldots+{}^kC_{m-2}\alpha^{k-m+2}x_m \\
x_3' = & \qquad\qquad\qquad\qquad \alpha^k x_3+{}^kC_1\alpha^{k-1}x_4+\ldots \\
 & \qquad\qquad\qquad\ldots+{}^kC_{m-3}\alpha^{k-m+3}x_m \\
\vdots & \qquad\qquad\qquad\qquad\vdots \qquad\qquad\qquad \vdots \\
x_m' = & \qquad\qquad\qquad\qquad\qquad\qquad \alpha^k x_m
\end{array}\right\},*$$

as is at once proved by induction.

For if we put

$$\alpha x_1+x_2 \text{ for } x_1,\ \ldots,\ \alpha x_{m-1}+x_m \text{ for } x_{m-1},\ \alpha x_m \text{ for } x_m$$

in $\quad \alpha^k x_1+{}^kC_1\alpha^{k-1}x_2+{}^kC_2\alpha^{k-2}x_3+\ldots+{}^kC_{m-1}\alpha^{k-m+1}x_m,$

we get, using $\qquad {}^{k+1}C_r={}^kC_r+{}^kC_{r-1},$

$$\alpha^{k+1}x_1+{}^{k+1}C_1\alpha^k x_2+{}^{k+1}C_2\alpha^{k-1}x_3+\ldots+{}^{k+1}C_{m-1}\alpha^{k-m+2}x_m.$$

Hence C is of finite order n, if and only if $m=1$ and α is a primitive n-th root of unity.

The inverse C^{-1} of C is at once verified to be

$$\left.\begin{array}{ll}
x_1' = & \alpha^{-1}x_1-\alpha^{-2}x_2+\alpha^{-3}x_3-\ldots+(-1)^{m+1}\alpha^{-m}x_m \\
x_2' = & \qquad\quad \alpha^{-1}x_2-\alpha^{-2}x_3+\ldots+(-1)^{m+2}\alpha^{-m+1}x_m \\
x_3' = & \qquad\qquad\qquad \alpha^{-1}x_3-\ldots+(-1)^{m+3}\alpha^{-m+2}x_m \\
\vdots & \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
x_m' = & \qquad\qquad\qquad\qquad\qquad\qquad \alpha^{-1}x_m
\end{array}\right\}.$$

* We suppose ${}^kC_t=0$ if $t>k$.

We can readily verify by induction, that C^{-k} is

$$\left.\begin{array}{rl}
x_1' = & \alpha^{-k}x_1 - {}^kC_{k-1}\alpha^{-k-1}x_2 + {}^{k+1}C_{k-1}\alpha^{-k-2}x_3 - {}^{k+2}C_{k-1}\alpha^{-k-3}x_4 + \dots \\
 & \quad \dots + (-1)^{m+1}\, {}^{k+m-2}C_{k-1}\alpha^{-k-m+1}x_m \\
x_2' = & \alpha^{-k}x_2 - {}^kC_{k-1}\alpha^{-k-1}x_3 + {}^{k+1}C_{k-1}\alpha^{-k-2}x_4 - \dots \\
 & \quad \dots + (-1)^{m}\, {}^{k+m-3}C_{k-1}\alpha^{-k-m+2}x_m \\
x_3' = & \alpha^{-k}x_3 - {}^kC_{k-1}\alpha^{-k-1}x_4 + \dots \\
 & \quad \dots + (-1)^{m-1}\, {}^{k+m-4}C_{k-1}\alpha^{-k-m+3}x_m \\
\vdots & \qquad \vdots \\
x_m' = & \alpha^{-k}x_m
\end{array}\right\}.$$

§ 9. Canonical Substitution.

We now prove the important theorem that, given any substitution A of degree m, we can find a substitution S such that $S^{-1}AS$ is the direct product of substitutions of the type C of § 8. On this account we shall call such a direct product a *canonical substitution.*

The theorem is true when $m = 1$. We shall assume it true for all substitutions of degree less than m, and prove that on this assumption it is true for any substitution A of degree m. The result will then follow at once by induction. By § 6 A can be transformed so that $(1, 0, 0, \dots, 0)$ is a pole of the transform of A. Suppose this transform is

$$\left.\begin{array}{l} x_1' = b_{11}x_1 + b_{12}x_2 + \dots + b_{1m}x_m \\ x_t' = \qquad\quad b_{t2}x_2 + \dots + b_{tm}x_m \ (t = 2, 3, \dots, m) \end{array}\right\}.$$

By the assumption A can be transformed so that

$$x_t' = b_{t2}x_2 + \dots + b_{tm}x_m \ (t = 2, 3, \dots, m)$$

is in canonical form.

Suppose that now A has been transformed into

$$\left.\begin{array}{l} x_1' = \alpha x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 \\ x_2' = \beta x_2 + x_3,\ x_3' = \beta x_3 + x_4,\ x_4' = \beta x_4,\ x_5' = \beta x_5 \end{array}\right\}, \dots \text{(i)}$$

to take a simple example. The method is general.

(*a*) Express this substitution in terms of new independent variables $\xi_1, x_2, x_3, x_4, x_5$, where

$$\xi_1 = x_1 + b_2x_2 + b_3x_3 + b_4x_4.$$

We then get

$$\begin{aligned}
\xi_1' = x_1' + b_2x_2' + b_3x_3' + b_4x_4' &= \alpha x_1 + (a_2 + \beta b_2)\,x_2 \\
&\quad + (a_3 + b_2 + \beta b_3)\,x_3 + (a_4 + b_3 + \beta b_4)\,x_4 + a_5x_5 \\
&= \alpha\xi_1 + (a_2 + \overline{\beta - \alpha}\, b_2)\,x_2 + (a_3 + b_2 + \overline{\beta - \alpha}\, b_3)\,x_3 \\
&\quad + (a_4 + b_3 + \overline{\beta - \alpha}\, b_4)\,x_4 + a_5x_5.
\end{aligned}$$

If then we choose b_2, b_3, b_4 so that

$$a_2+\overline{\beta-\alpha}b_2 = a_3+b_2+\overline{\beta-\alpha}b_3 = a_4+b_3+\overline{\beta-\alpha}b_4 = 0,$$

A will be the direct product of a substitution on x_2, x_3, x_4, and one on x_1, x_5. Each of these can be transformed into a canonical substitution by the assumption. Hence, since a change of variables is equivalent to a transformation (§ 5), the result is established.

In the above we assumed $\beta \neq \alpha$. It is still necessary to discuss the case in which A has only one distinct characteristic-root.

Suppose that A were transformed not into (i) but into

$$\left.\begin{array}{l} x_1' = \alpha x_1+a_2x_2+a_3x_3+a_4x_4+a_5x_5+a_6x_6+a_7x_7+a_8x_8 \\ x_2' = \alpha x_2+x_3,\ x_3' = \alpha x_3+x_4,\ x_4' = \alpha x_4; \\ \qquad x_5' = \alpha x_5+x_6,\ x_6' = \alpha x_6+x_7,\ x_7' = \alpha x_7;\ x_8' = \alpha x_8 \end{array}\right\},$$

taking again a simple example, but the method is general. It will be noticed that the constituents of the direct product into which

$$x_t' = b_{t2}x_2+ \ldots +b_{tm}x_m \ (t = 2, 3, \ldots, m)$$

was transformed are arranged so that the constituent with the greatest number of variables comes first, that with the next greatest number second, and so on.

There are two cases to consider.

(*b*) First suppose $a_2 = 0$.

Putting $\xi_1 = x_1-a_3x_2-a_4x_3$ the substitution becomes

$$\left.\begin{array}{l} \xi_1' = \alpha\xi_1+a_5x_5+a_6x_6+a_7x_7+a_8x_8,\ x_2' = \alpha x_2+x_3, \\ \qquad\qquad\qquad\qquad\qquad\qquad x_3' = \alpha x_3+x_4 \\ x_4' = \alpha x_4;\ x_5' = \alpha x_5+x_6,\ x_6' = \alpha x_6+x_7,\ x_7' = \alpha x_7; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad x_8' = \alpha x_8 \end{array}\right\}.$$

This is the direct product of a substitution on x_2, x_3, x_4 and one on $\xi_1, x_5, x_6, x_7, x_8$. Each of these can be transformed into canonical form by our assumption.

(*c*) Secondly, suppose $a_2 \neq 0$.

Putting *

$$\begin{aligned} \xi_2 &= a_2x_2+a_3x_3+a_4x_4+a_5x_5+a_6x_6+a_7x_7+a_8x_8, \\ \xi_3 &= \qquad a_2x_3+a_3x_4 \qquad +a_5x_6+a_6x_7, \\ \xi_4 &= \qquad\qquad a_2x_4 \qquad\qquad +a_5x_7, \end{aligned}$$

* The reason of this transformation will be clear in Ch. V.

we get

$$\left.\begin{array}{l} x_1' = \alpha x_1 + \xi_2,\ \xi_2' = \alpha\xi_2 + \xi_3,\ \xi_3' = \alpha\xi_3 + \xi_4,\ \xi_4' = \alpha\xi_4 \\ x_5' = \alpha x_5 + x_6,\ x_6' = \alpha x_6 + x_7,\ x_7' = \alpha x_7; \qquad x_8' = \alpha x_8 \end{array}\right\},$$

which is in canonical form.

It will be noticed that we have transformed A into canonical form by a series of successive transformations. This is equivalent to transformation by a single substitution (§ 5).

In practice it is convenient to first transform A into a *normal* substitution (§ 6).

Corollary I.

A substitution A of degree m is transformable into a multiplication if and only if it has m poles

$$(X_{11}, X_{21}, \ldots, X_{m1}), (X_{12}, X_{22}, \ldots, X_{m2}), \ldots, (X_{1m}, X_{2m}, \ldots, X_{mm})$$

such that

$$\begin{vmatrix} X_{11} & X_{12} & \cdot & \cdot & \cdot & X_{1m} \\ X_{21} & X_{22} & \cdot & \cdot & \cdot & X_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ X_{m1} & X_{m2} & \cdot & \cdot & \cdot & X_{mm} \end{vmatrix} \neq 0.$$

First suppose the poles exist, and let P be the substitution

$$x_t' = X_{t1}x_1 + X_{t2}x_2 + \ldots + X_{tm}x_m \ (t = 1, 2, \ldots, m).$$

Let the poles correspond respectively to the characteristic-roots $\lambda_1, \lambda_2, \ldots, \lambda_m$ of A.

Then we verify at once, using

$$\lambda_i X_{ti} = a_{t1}X_{1i} + a_{t2}X_{2i} + \ldots + a_{tm}X_{mi},$$

that $PA = MP$, M being the multiplication

$$x_1' = \lambda_1 x_1,\ x_2' = \lambda_2 x_2,\ \ldots,\ x_m' = \lambda_m x_m.$$

Hence P^{-1} transforms A into the multiplication M.

Conversely, suppose $P^{-1}MP = A$. Then M has the poles

$$(1, 0, 0, \ldots, 0, 0), (0, 1, 0, \ldots, 0, 0), \ldots, (0, 0, 0, \ldots, 0, 1).$$

Therefore the corresponding poles of A are (§ 5)

$$(p_{11}, p_{21}, \ldots, p_{m1}), (p_{12}, p_{22}, \ldots, p_{m2}), \ldots, (p_{1m}, p_{2m}, \ldots, p_{mm});$$

and the determinant formed by these poles is not zero, since it is the determinant of P'.

Corollary II.

A substitution of finite order is transformable into a multiplication whose coefficients are roots of unity.

For a substitution has the same order as any transform (§ 5), and therefore the same order as its canonical form. But by § 8 the canonical form is not of finite order unless it is a multiplication.

Ex. 1. Transform

$$(x-2y-2z,\ 2x+3y+4z,\ -2x-y-2z)$$

into a multiplication.

[The poles are $(1, 1, -1)$, $(0, 1, -1)$, $(-2, 0, 1)$ corresponding respectively to the characteristic-roots 1, -1, 2.

Therefore by Corollary I the inverse of $(x-2z, x+y, -x-y+z)$, i.e. $(x+2y+2z, -x-y-2z, y+z)$ transforms the given substitution into $(x, -y, 2z)$.]

Ex. 2. Transform into multiplications the substitutions of § 6, Ex. 3, 4 (i), 5 (i) and (ii), 6 to 10.

Ex. 3. Transform

$A \equiv (x-y+2z-2w,\ -3y-2w,\ -2x+2y-3z+3w,\ 2y+w)$

into canonical form.

[By § 6, Ex. 15, $K \equiv (x, y, x+z, y+w)$ transforms A into normal form $\quad (-x+y+2z-2w,\ -y-2w,\ -z+w,\ -w)$.

Considered as a substitution on z, w, it is already in canonical form. Considered as a substitution on y, z, w, it is the type (b) of § 9. We transform by $L \equiv (x, y+2z, z, w)$ and obtain $(-x+y-2w, -y, -z+w, -w)$.

This is again of type (b). We transform by

$M \equiv (x+2z, y, z, w)$, and get $(-x+y, -y, -z+w, -w)$.

Hence $LM \equiv (x+2z, y+2z, z, w)$ transforms

$$(-x+y+2z-2w,\ -y+2w,\ -z+w,\ -w)$$

into canonical form.

Hence A is transformed into the canonical substitution

$$(-x+y,\ -y,\ -z+w,\ -w),$$

by $\quad KLM \equiv (3x+2z,\ 2x+y+2z,\ x+z,\ y+w)$.]

Ex. 4. Transform

$$A \equiv (x+2y-2z+2w,\ y+z-2w,\ 2z-w,\ 2w)$$

into canonical form.

[A is in normal form.

Applying § 9 (c), put $\omega = -w$. Then A becomes

$$(x+2y-2z-2\omega,\ y+z+2\omega,\ 2z+\omega,\ 2\omega),$$

Applying § 9 (*a*), put $y = Y + az + b\omega$. Then

$$Y' = y' - az' - b\omega' = Y + (1-a)z + (2-b)\omega.$$

Taking therefore $a = 1$ and $b = 2$, put $y = Y + z + 2\omega$, and A becomes $(x + 2Y + 2\omega,\ Y,\ 2z + \omega,\ 2\omega)$.

Put now as before $x = \xi + cz + d\omega$, and we find that we must take $c = 0$, $d = 2$. Then A becomes $(\xi + 2Y,\ Y,\ 2z + \omega,\ 2\omega)$. Finally, put $2Y = \eta$, and A becomes $(\xi + \eta,\ \eta,\ 2z + \omega,\ 2\omega)$, where $\xi = x + 2w$, $\eta = 2y - 2z + 4w$, $z = z$, $\omega = -w$. Therefore A is transformed into canonical form by

$$(x + 2w,\ 2y - 2z + 4w,\ z,\ -w).$$

Ex. 3 and 4 illustrate the process given in § 9 for transforming A into a canonical substitution. A method which is perhaps easier in practice is given in Ch. II, § 5, Corollary III.]

Ex. 5. Transform into canonical form

$$(3x + y - 2z,\ 2x + 3y - 3z,\ 4x + 2y - 3z),$$

$$(3x + 3y + 10z + 10w,\ 4x - y + 4z + 6w,\ -2x + 2y + z - w,\ -2y - 4z - 3w),$$

and

$$(-2x_1 - x_2 - x_3 + 3x_4 + 2x_5,\ -4x_1 + x_2 - x_3 + 3x_4 + 2x_5,\ x_1 + x_2 - 3x_4 - 2x_5,\ -4x_1 - 2x_2 - x_3 + 5x_4 + x_5,\ 4x_1 + x_2 + x_3 - 3x_4).$$

[$(x + y,\ y + z,\ z)$, $(x + y,\ y,\ -z + w,\ -w)$, and $(-x_1 + x_2,\ -x_2,\ 2x_3 + x_4,\ 2x_4,\ 2x_5)$.*]

Ex. 6. Transform

$$(ax_1 + bx_2 + cx_3 + dx_4 + \dots,\ ax_2 + bx_3 + cx_4 + \dots,\ ax_3 + bx_4 + \dots,\ \dots,\ ax_m)$$

into canonical form if $a \neq 0$, $b \neq 0$.

[Using (*c*) of § 9, we get as the transforming substitution

$$(x_1,\ bx_2 + cx_3 + dx_4 + ex_5 + \dots,\ b_2x_3 + c_2x_4 + d_2x_5 + \dots,\ b_3x_4 + c_3x_5 + \dots,\ b_4x_5 + \dots,\ \dots,\ b_{m-1}x_m),$$

where $(b + cx + dx^2 + ex^3 + \dots)^r \equiv b_r + c_rx + d_rx^2 + e_rx^3 + \dots$.]

Ex. 7. Show that, if a linear substitution of degree m and prime order p has integral coefficients, m is a multiple of $p-1$.

[The characteristic-equation $\theta(\lambda) = 0$ has one root which is a p-th root of unity. Hence the H. C. F. of $\theta(\lambda) = 0$ and $\lambda^{p-1} + \lambda^{p-2} + \dots + \lambda + 1$ is not unity, and must therefore be $\lambda^{p-1} + \lambda^{p-2} + \dots + \lambda + 1$.]

Discuss the case in which p is not prime.

Ex. 8. Show that the most general form of substitution of order 2 and degree m with characteristic-roots $-1, 1, 1, \dots, 1$ is

$$x_t' = x_t + k_t(c_1x_1 + c_2x_2 + \dots + c_mx_m),$$

where $$0 = 2 + k_1c_1 + k_2c_2 + \dots + k_mc_m.$$

[Transform $(-x_1,\ x_2,\ x_3,\ \dots,\ x_m)$ by any substitution, and we get a substitution of the given type.]

* See Burnside, *Proc. London Math. Soc.*, xxx (1899), p. 194.

§ 10. Symmetric, Orthogonal, ..., Substitutions.

Let A be the substitution

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \ (t = 1, 2, \ldots, m).$$

If all the coefficients a_{ij} of A are real quantities, A is called *real.*

If for all values of i and j $a_{ij} = a_{ji}$, A is called *symmetric.* The determinant of A is symmetrical about its leading diagonal.

If $a_{ij} = -a_{ji}$ (so that $a_{ii} = 0$), A is called *alternate* and its determinant *skew-symmetric.*

If $a_{ij} = \bar{a}_{ji}$ (so that a_{ii} is real), A is called *Hermitian.* If the Hermitian form $\Sigma a_{ij}\bar{x}_i x_j$* is definite (Ch. III, § 2), A is called a *definite Hermitian* substitution.†

If $AA' = E$, A is called *orthogonal.* Forming the product AA' we see that we must have

$$\left.\begin{aligned} a_{1i}^2 + a_{2i}^2 + a_{3i}^2 + \ldots + a_{mi}^2 &= 1 \\ a_{1i}a_{1j} + a_{2i}a_{2j} + \ldots + a_{mi}a_{mj} &= 0 \ (i \neq j) \end{aligned}\right\} \ldots\ldots\ldots\ldots (\alpha)$$

Since A' is the inverse of A, we have also $A'A = E$, and therefore

$$\left.\begin{aligned} a_{i1}^2 + a_{i2}^2 + a_{i3}^2 + \ldots + a_{im}^2 &= 1 \\ a_{i1}a_{j1} + a_{i2}a_{j2} + \ldots + a_{im}a_{jm} &= 0 \ (i \neq j) \end{aligned}\right\} \ldots\ldots\ldots (\alpha')$$

The equations (α') are, of course, algebraically equivalent to (α) and vice versa.

If $A\bar{A}' = E$, A is called *unitary.* As in the case of an orthogonal substitution, we have

$$\left.\begin{aligned} a_{1i}\bar{a}_{1i} + a_{2i}\bar{a}_{2i} + \ldots + a_{mi}\bar{a}_{mi} &= 1 \\ a_{1i}\bar{a}_{1j} + a_{2i}\bar{a}_{2j} + \ldots + a_{mi}\bar{a}_{mj} &= 0 \ (i \neq j) \end{aligned}\right\} \ldots\ldots\ldots (\beta)$$

and

$$\left.\begin{aligned} a_{i1}\bar{a}_{i1} + a_{i2}\bar{a}_{i2} + \ldots + a_{im}\bar{a}_{im} &= 1 \\ a_{i1}\bar{a}_{j1} + a_{i2}\bar{a}_{j2} + \ldots + a_{im}\bar{a}_{jm} &= 0 \ (i \neq j) \end{aligned}\right\} \ldots\ldots\ldots (\beta')$$

If A is *real,* evidently A^{-1} is real; and the product of two or more real substitutions is real.

If A is *symmetric,* so is A^{-1}, as is at once evident from § 3. The product of two symmetric substitutions is not, however, in general symmetric.

* In $\Sigma a_{ij}\bar{x}_i x_j$ i and j take independently all possible values from 1 to m; so that $\Sigma a_{ij}\bar{x}_i x_j$ contains m^2 terms.

† Some authors confine the title 'Hermitian' to a 'definite Hermitian' substitution.

Similarly if A is *alternate* or *Hermitian*.*

If A is *orthogonal*, so is A^{-1}. For

$$A^{-1}(A^{-1})' = A^{-1}A'^{-1} = (A'A)^{-1} = E.$$

The product of two or more orthogonal substitutions is orthogonal. For instance, if $AA' = E$ and $BB' = E$,

$$(AB)(AB)' = AB \,.\, B'A' = AA' = E.$$

Similarly if A is *unitary*.

The transform of an orthogonal substitution by an orthogonal substitution is orthogonal.

For if S and A are orthogonal, S^{-1} is orthogonal; and therefore $S^{-1}AS$ must be orthogonal, being the product of the three orthogonal substitutions S^{-1}, A, S.

Similarly,

The transform of a unitary substitution by a unitary substitution is unitary.

If A is *symmetric* and B is any substitution, $B'AB$ is symmetric.

For the elements in the i-th row and j-th column and in the j-th row and i-th column of the matrix of $B'AB$ are respectively (§ 2)

$$\sum_{\sigma,\tau} b_{i\sigma}a_{\sigma\tau}b_{j\tau} \text{ and } \sum_{\sigma,\tau} b_{j\sigma}a_{\sigma\tau}b_{i\tau} = \sum_{\sigma,\tau} b_{j\sigma}a_{\tau\sigma}b_{i\tau} = \sum_{\sigma,\tau} b_{i\tau}a_{\tau\sigma}b_{j\sigma},$$

which are identical.

If B is orthogonal, $B' = B^{-1}$, and hence:—

The transform of a symmetric substitution by an orthogonal substitution is symmetric.

Similarly, if A is *Hermitian*, $\bar{B}'AB$ is Hermitian, and

The transform of a Hermitian substitution by a unitary substitution is Hermitian.

Similarly, if A is *alternate*, $B'AB$ is alternate, and

The transform of an alternate substitution by an orthogonal substitution is alternate.

If a substitution A has two of the properties (i) *being real,* (ii) *being orthogonal,* (iii) *being unitary, it has the third.*

For example, suppose A is real and orthogonal, then $A' = \bar{A}'$ and $AA' = E$, therefore $A\bar{A}' = E$.

* We assume that the determinant of the substitution does not vanish, which cannot be the case for an alternate substitution of *odd* degree.

Similarly,

If A has two of the properties (i) *being symmetric,* (ii) *being orthogonal,* (iii) *being of order* 2, *it has the third.*

For example, suppose A is symmetric and of order 2. Then $A = A'$ and $A^2 = E$. Therefore $AA' = E$.

Similarly,

If A has two of the properties (i) *being Hermitian,* (ii) *being unitary,* (iii) *being of order* 2, *it has the third.*

Ex. 1. If A is orthogonal, the determinant of A is ± 1.

[The determinant of AA' = the product of the determinants of A and A' = the square of the determinant of A. Therefore the square of this determinant = the determinant of $E = 1$.]

Ex. 2. If A is unitary, the determinant of A has unit modulus.

[As in Ex. 1.]

Ex. 3. If $(Y_1, Y_2, \ldots, Y_m)$ is the pole of $S^{-1}AS$ corresponding to the pole $(X_1, X_2, \ldots, X_m)$ of A, and S is orthogonal,

then $$Y_1^2 + Y_2^2 + \ldots + Y_m^2 = X_1^2 + X_2^2 + \ldots + X_m^2.$$

[Square and add equations (ii) of § 6.]

Ex. 4. If S is unitary in Ex. 3,

$$Y_1 \bar{Y}_1 + Y_2 \bar{Y}_2 + \ldots + Y_m \bar{Y}_m = X_1 \bar{X}_1 + X_2 \bar{X}_2 + \ldots + X_m \bar{X}_m.$$

Ex. 5. If (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) are the direction-cosines of three mutually perpendicular lines, the substitution whose matrix is $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$ is orthogonal.

As an example, consider

$$(\tfrac{1}{3}x + \tfrac{2}{3}y + \tfrac{2}{3}z,\ \tfrac{2}{3}x + \tfrac{1}{3}y - \tfrac{2}{3}z,\ \tfrac{2}{3}x - \tfrac{2}{3}y + \tfrac{1}{3}z).$$

Ex. 6. The most general orthogonal substitution of degree 2 is

$$(\cos\omega . x - \sin\omega . y,\ \ \pm\sin\omega . x \pm \cos\omega . y).$$

Ex. 7. The substitution obtained from a given orthogonal substitution by permuting any two rows or columns in its matrix is also orthogonal.

Similarly for a unitary substitution.

Ex. 8. A multiplication is unitary if its coefficients have unit modulus.

Ex. 9. The substitution of § 3, Ex. 7 is orthogonal.

[It is symmetric and of order 2.]

Ex. 10. Prove that the substitution with matrix

$$\left|\begin{matrix} \dfrac{1+\lambda^2-\mu^2-\nu^2}{\kappa} & \dfrac{2(\lambda\mu+\nu)}{\kappa} & \dfrac{2(\nu\lambda-\mu)}{\kappa} \\[2ex] \dfrac{2(\lambda\mu-\nu)}{\kappa} & \dfrac{1-\lambda^2+\mu^2-\nu^2}{\kappa} & \dfrac{2(\mu\nu+\lambda)}{\kappa} \\[2ex] \dfrac{2(\nu\lambda+\mu)}{\kappa} & \dfrac{2(\mu\nu-\lambda)}{\kappa} & \dfrac{1-\lambda^2-\mu^2+\nu^2}{\kappa} \end{matrix}\right|,$$

where $$\kappa = 1+\lambda^2+\mu^2+\nu^2$$
is orthogonal; and find its characteristic-roots and poles.

[1 and $\dfrac{1}{\kappa}(2-\kappa\pm 2\sqrt{1-\kappa})$;

$(\lambda,\ \mu,\ \nu)$ and $(\lambda\nu\mp\mu\sqrt{1-\kappa},\ \mu\nu\pm\lambda\sqrt{1-\kappa},\ -\lambda^2-\mu^2)$.]

Ex. 11. If $x_t' = a_t x_1 + b_t x_2 + c_t x_3 + d_t x_4 + e_t x_5$ $(t = 1, 2, 3, 4, 5)$ is orthogonal, any two of the five spheres

$$(d_t + ie_t)(x^2+y^2+z^2) + 2R(a_t x + b_t y + c_t z) + (-d_t + ie_t)R^2 = 0$$

cut orthogonally.

Show how to obtain similarly a system of four mutually orthogonal circles in a plane.

Find the circles when the orthogonal substitution is

$$\left(x_1 + \frac{1}{2}x_3 - \frac{i}{2}x_4,\ x_2 + \frac{1}{2}x_3 - \frac{i}{2}x_4,\ x_1 + x_2 - \frac{3}{4}x_3 - \frac{5i}{4}x_4,\right.$$
$$\left. ix_1 + ix_2 - \frac{i}{4}x_3 + \frac{7}{4}x_4\right).$$

Ex. 12. If $(ax_1 + bx_2 + cx_3 + dx_4 + \dots,\ ax_2 + bx_3 + cx_4 + \dots,$
$ax_3 + bx_4 + \dots,\ \dots)$
is orthogonal, $a^2 = 1$, $b = c = d = \dots = 0$; and if it is unitary, $|a| = 1$, $b = c = d = \dots = 0$.

Ex. 13. If the real symmetric substitution A transforms the unitary substitution U into another unitary substitution, U is permutable with A^2. Similarly if U and $A^{-1}UA$ are orthogonal.

[$A^{-1}UA \,.\, \overline{A^{-1}UA}' = E$ gives $UA^2 = A^2U$, since $A = A' = \bar{A}$ and $U\bar{U}' = E$.]

Ex. 14. The characteristic-roots of a unitary substitution have unit modulus.

[If $(X_1, X_2, \dots, X_m)$ is a pole of A,

$$\begin{aligned} \Sigma\lambda\bar{\lambda}X_i\bar{X}_i &= \Sigma(a_{i1}X_1 + a_{i2}X_2 + \dots + a_{im}X_m)(\bar{a}_{i1}\bar{X}_1 + \bar{a}_{i2}\bar{X}_2 + \dots + \bar{a}_{im}\bar{X}_m) \\ &= \Sigma X_i\bar{X}_i. \quad \therefore \quad \lambda\bar{\lambda} = 1. \end{aligned}$$

Ex. 15. If $(X_1, X_2, \ldots, X_m)$ is a pole of an orthogonal substitution corresponding to a characteristic-root other than ± 1,

$$X_1^2 + X_2^2 + \ldots + X_m^2 = 0.$$

[As in the last Example.]

Ex. 16. If $(X_1, X_2, \ldots, X_m)$, $(Y_1, Y_2, \ldots, Y_m)$ are poles of a symmetric substitution corresponding to distinct characteristic-roots, $X_1Y_1 + X_2Y_2 + \ldots + X_mY_m = 0$. What is the similar theorem for a Hermitian substitution?

Ex. 17. Express $(\alpha x_1 + x_2, \alpha x_2 + x_3, \ldots, \alpha x_{m-1} + x_m, \alpha x_m)$ as the product of two symmetric substitutions.

[$(x_m, \alpha x_{m-1} + x_m, \ldots, \alpha x_2 + x_3, \alpha x_1 + x_2)$
and $(x_m, x_{m-1}, \ldots, x_2, x_1.$]

Ex. 18. Find a symmetric substitution of degree 2 with $(1, i)$ as pole. Show that its characteristic-roots are equal, and that it cannot be transformed into a multiplication unless it is a similarity.

[The most general substitution is

$$(2ax + (a-b)iy,\ (a-b)ix + 2by).]$$

Ex. 19. The coefficients of the characteristic-equation of a Hermitian substitution are real.

Ex. 20. How far will the definitions and results of § 10 hold if the determinant $|a|$ of A is zero?

§ 11. Invariants.

Suppose $f(x_1, x_2, \ldots, x_m)$ a function of $x_1, x_2, \ldots, x_m$ such that

$$f(x_1, x_2, \ldots, x_m) \equiv f(a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m,$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m, \ldots, a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mm}x_m)$$

for all values of $x_1, x_2, \ldots, x_m$.

Then $f(x_1, x_2, \ldots, x_m)$ is called an *invariant* of the substitution A

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \ (t = 1, 2, \ldots, m).$$

The equations (α) of § 10 show that

$$x_1^2 + x_2^2 + \ldots + x_m^2$$

is an invariant of A, if and only if A is orthogonal.

If $f(x_1, x_2, \ldots, x_m)$ is an invariant of A and of B, it is evidently an invariant of AB. Hence the product of orthogonal substitutions is orthogonal, as proved before.

If $PAP^{-1} = B$, and $f(x_1, x_2, \ldots, x_m)$ is an invariant of A, then

$$f(p_{11}x_1 + p_{12}x_2 + \ldots + p_{1m}x_m, p_{21}x_1 + p_{22}x_2 + \ldots + p_{2m}x_m, \ldots, p_{m1}x_1 + p_{m2}x_2 + \ldots + p_{mm}x_m)$$

is an invariant of B.

Since $P^{-1}BP = A$, the equations

$$\left.\begin{aligned} x_t' &= b_{t1}x_1 + b_{t2}x_2 + \ldots + b_{tm}x_m \\ y_t' &= p_{t1}x_1' + p_{t2}x_2' + \ldots + p_{tm}x_m' \\ y_t &= p_{t1}x_1 + p_{t2}x_2 + \ldots + p_{tm}x_m \end{aligned}\right\} (t = 1, 2, \ldots, m)$$

give by § 5

$$y_t' = a_{t1}y_1 + a_{t2}y_2 + \ldots + a_{tm}y_m \quad (t = 1, 2, \ldots, m).$$

Hence $\qquad f(y_1', y_2', \ldots, y_m') \equiv f(y_1, y_2, \ldots, y_m)$

or

$$f(p_{11}x_1' + p_{12}x_2' + \ldots + p_{1m}x_m', \ldots, p_{m1}x_1' + \ldots + p_{mm}x_m') \equiv f(p_{11}x_1 + \ldots + p_{1m}x_m, \ldots, p_{m1}x_1 + \ldots + p_{mm}x_m),$$

where $\qquad x_t' = b_{t1}x_1 + b_{t2}x_2 + \ldots + b_{tm}x_m,$

which proves the result.

If

$$f(x_1', \ldots, x_m', \bar{x}_1', \ldots, \bar{x}_m') \equiv f(x_1, \ldots, x_m, \bar{x}_1, \ldots, \bar{x}_m),$$

where

$$\left.\begin{aligned} x_t' &= a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \\ \bar{x}_t' &= \bar{a}_{t1}\bar{x}_1 + \bar{a}_{t2}\bar{x}_2 + \ldots + \bar{a}_{tm}\bar{x}_m \end{aligned}\right\} (t = 1, 2, \ldots, m), \ldots\ldots \text{(i)}$$

$f(x_1, \ldots, x_m, \bar{x}_1, \ldots, \bar{x}_m)$ is usually called an invariant of A; though perhaps it would be more correct to call it an invariant of the substitution of degree $2m$ on $x_1, \ldots, x_m, \bar{x}_1, \ldots, \bar{x}_m$ defined by the equations (i).

The equations (β) of § 10 show that

$$x_1\bar{x}_1 + x_2\bar{x}_2 + \ldots + x_m\bar{x}_m$$

is an invariant of A, if and only if A is unitary.

Since $PAP^{-1} = B$ gives $\bar{P}\bar{A}\bar{P}^{-1} = \bar{B}$, we get by considering $f(x_1, \ldots, x_m, \bar{x}_1, \ldots, \bar{x}_m)$ as an invariant of the substitution (i) of degree $2m$:—

If $PAP^{-1} = B$ and $f(x_1, \ldots, x_m, \bar{x}_1, \ldots, \bar{x}_m)$ is an invariant of A, then

$$f(p_{11}x_1 + \ldots + p_{1m}x_m, \ldots, p_{m1}x_1 + \ldots + p_{mm}x_m, \bar{p}_{11}\bar{x}_1 + \ldots + \bar{p}_{1m}\bar{x}_m, \ldots, \bar{p}_{m1}\bar{x}_1 + \ldots + \bar{p}_{mm}\bar{x}_m)$$

is an invariant of B.

If

$$f(a_{11}x_1+a_{12}x_2+\ldots+a_{1m}x_m,\ a_{21}x_1+a_{22}x_2+\ldots+a_{2m}x_m,$$
$$\ldots,\ a_{m1}x_1+a_{m2}x_2+\ldots+a_{mm}x_m) \equiv k \,.\, f(x_1, x_2, \ldots, x_m),$$

for all values of $x_1, x_2, \ldots, x_m$ (where k does not involve $x_1, x_2, \ldots, x_m$), then $f(x_1, x_2, \ldots, x_m)$ is sometimes called a *relative invariant* of A.

An invariant for which $k=1$ is then called an *absolute invariant*. As before explained, the word 'invariant' will denote an 'absolute invariant' unless the contrary is stated.

Ex. 1. $\alpha x_1x_4-\alpha^{-1}x_3x_2$ is an invariant of

$$(\alpha x_1+x_2,\ \alpha x_2,\ \alpha^{-1}x_3+x_4,\ \alpha^{-1}x_4).$$

[If we operate with the substitution on $\alpha x_1x_4-\alpha^{-1}x_3x_2$ we get $\alpha(\alpha x_1+x_2)(\alpha^{-1}x_4)-\alpha^{-1}(\alpha^{-1}x_3+x_4)(\alpha x_2) \equiv \alpha x_1x_4-\alpha^{-1}x_3x_2$.]

Ex. 2. (i) $y^2-yz-2xz$ is an invariant of $(x+y,\ y+z,\ z)$;

(ii) $x_3{}^2+x_3x_4+x_3x_5+2x_1x_5-2x_2x_4-3x_2x_5$

of $(-x_1+x_2,\ -x_2+x_3,\ -x_3+x_4,\ -x_4+x_5,\ -x_5)$;

(iii) $x_1y_2z_3-x_1y_3z_2+x_3y_1z_2-x_2y_1z_3+x_2y_3z_1-x_3y_2z_1$

and $-x_3y_2z_2+x_3y_1z_3+x_3y_3z_1+x_2y_3z_3$

of $x_1'=\alpha(x_1+x_2),\ x_2'=\alpha x_2,\ y_1'=\beta(y_1+y_2),\ y_2'=\beta y_2,$

$z_1'=\gamma(z_1+z_2),\ z_2'=\gamma z_2,$

where $\alpha\beta\gamma=1$;

(iv) $-2x_2x_5+x_2x_6+x_3x_5+2x_1x_6+2x_3x_4$

of $(x_1+x_2,\ x_2+x_3,\ x_3,\ x_4+x_5,\ x_5+x_6,\ x_6)$.

Ex. 3. The substitution $(\alpha x+y,\ \alpha y)$ has no algebraic invariant other than $a_0+a_1y+a_2y^2+a_3y^3+\ldots$. In what cases is this an invariant?

Ex. 4. The number of linearly independent algebraic invariants of order k of A is equal to the number of such invariants of any transform of A.

Ex. 5. Find the linear relative invariants of A.

[$Z_1x_1+Z_2x_2+\ldots+Z_mx_m$; where $(Z_1, Z_2, \ldots, Z_m)$ is a pole of A'.]

Ex. 6. Find the conditions that $x_1{}^2+x_2{}^2+\ldots+x_m{}^2$ should be a relative invariant of A.

Ex. 7. If A is orthogonal, $\Sigma p_{ij}x_ix_j\,(i, j=1, 2, \ldots, m)$ is an invariant of BAB^{-1}; where P is the substitution BB'.

Ex. 8. If mod. $\alpha=1$, $\alpha x_1\bar{x}_2-\bar{\alpha}x_2\bar{x}_1$ is an invariant of $(\alpha x_1+x_2,\ \alpha x_2)$.

[If we operate with the given substitution on $\alpha x_1\bar{x}_2-\bar{\alpha}x_2\bar{x}_1$) we get

$$\alpha(\alpha x_1+x_2)(\bar{\alpha}\bar{x}_2)-\bar{\alpha}(\alpha x_2)(\bar{\alpha}\bar{x}_1+\bar{x}_2)=\alpha\bar{\alpha}(\alpha x_1\bar{x}_2-\bar{\alpha}x_2\bar{x}_1)$$
$$=\alpha x_1\bar{x}_2-\bar{\alpha}x_2\bar{x}_1.]$$

Ex. 9. If

$$\alpha\bar{\beta}=1,\ \tfrac{1}{2}x_2(\bar{y}_2-\alpha\bar{y}_3)+\tfrac{1}{2}\bar{y}_2(x_2-\alpha^{-1}x_3)-\alpha^2x_1\bar{y}_3-\alpha^{-2}\bar{y}_1x_3$$

is an invariant of

$$x_1'=\alpha x_1+x_2,\ x_2'=\alpha x_2+x_3,\ x_3'=\alpha x_3,\ y_1'=\beta y_1+y_2,\ y_2'=\beta y_2+y_3.\ y_3'=\beta y_3.$$

Ex. 10. A substitution with

$$x_1^2+x_2^2+\ldots+x_m^2 \quad\text{and}\quad \lambda_1x_1\bar{x}_1+\lambda_2x_2\bar{x}_2+\ldots+\lambda_mx_m\bar{x}_m$$

as invariants, where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are real, is a real orthogonal substitution.

[See *Proc. London Math. Soc.*, 2, xii, p. 92.]

Ex. 11. The two theorems of § 11 hold for relative invariants as well as for absolute invariants.

Ex. 12. Trace the connexion between invariants or covariants of a quantic and relative invariants of homogeneous linear substitutions.

[Suppose the quantic $a_0x^2+2a_1xy+a_2y^2$ becomes

$$a_0'x'^2+2a_1'x'y'+a_2'y'^2$$

when we put $x=l_1x'+m_1y'$, $y=l_2x'+m_2y'$. Then any covariant of the quantic may be considered as a relative invariant of the substitution in the variables a_0, a_1, a_2, x, y,

$$\left.\begin{array}{l} a_0'=l_1^2a_0+2l_1l_2a_1+l_2^2a_2,\ a_1'=l_1m_1a_0+(l_1m_2+l_2m_1)a_1+l_2m_2a_2 \\ a_2'=m_1^2a_0+2m_1m_2a_1+m_2^2a_2,\quad x=l_1x'+m_1y',\quad y=l_2x'+m_2y' \end{array}\right\}.$$

Show that the determinant of this substitution is a power of the determinant of the substitution $x'=l_1x+m_1y$, $y'=l_2x+m_2y$; and extend the result to the general case of a q-ary p-ic.]

§ 12. Hermitian, Real Symmetric, or Real Alternate Substitution transformable into a Multiplication.

A Hermitian substitution can be transformed into a multiplication by a unitary substitution.

This is true when the degree m of the Hermitian substitution A

$$x_t'=a_{t1}x_1+a_{t2}x_2+\ldots+a_{tm}x_m \text{ (where } a_{ij}=\bar{a}_{ji})$$

is unity. We assume it true for all Hermitian substitutions of degree less than m. We then prove it holds true for A; and the result will follow by induction.

Let $(X_1, X_2, \ldots, X_m)$ be a pole of A. Since we are only concerned with the *ratios* of $X_1, X_2, \ldots, X_m$, we may suppose X_1 real, $X_1+1\neq 0$, and $X_1\bar{X}_1+X_2\bar{X}_2+\ldots+X_m\bar{X}_m=1$.

Consider the substitution P with matrix

$$\left|\begin{array}{ccccc} X_1 & \bar{X}_2 & \bar{X}_3 & \cdots & \bar{X}_m \\ X_2 & \dfrac{X_2\bar{X}_2}{X_1+1}-1 & \dfrac{X_2\bar{X}_3}{X_1+1} & \cdots & \dfrac{X_2\bar{X}_m}{X_1+1} \\ X_3 & \dfrac{X_3\bar{X}_2}{X_1+1} & \dfrac{X_3\bar{X}_3}{X_1+1}-1 & \cdots & \dfrac{X_3\bar{X}_m}{X_1+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ X_m & \dfrac{X_m\bar{X}_2}{X_1+1} & \dfrac{X_m\bar{X}_3}{X_1+1} & \cdots & \dfrac{X_m X_m}{X_1+1}-1 \end{array}\right| \quad \ldots\text{(i)}$$

We readily verify that, since X_1 is real and

$$X_1\bar{X}_1 + \ldots + X_m\bar{X}_m = 1,$$

the equations (β') of § 10 are satisfied and therefore P is unitary.*

Again, the pole of $P^{-1}AP$ corresponding to the pole $(X_1, X_2, \ldots, X_m)$ of A is $(1, 0, 0, \ldots, 0)$ by § 6 (ii).

Hence if $P^{-1}AP = B$, $b_{21} = b_{31} = \ldots = b_{m1} = 0$.

But since A is Hermitian and P unitary, B is Hermitian (§ 10); so that $b_{12} = b_{13} = \ldots = b_{1m} = 0$.

Hence $P^{-1}AP$ is of the form

$$x_1' = \lambda_1 x_1, \quad x_2' = b_{t2}x_2 + b_{t3}x_3 + \ldots + b_{tm}x_m \ (t = 2, 3, \ldots, m).$$

Now by our assumption we can find a unitary substitution Q transforming the Hermitian substitution

$$x_t' = b_{t2}x_2 + b_{t2}x_3 + \ldots + b_{tm}x_m \ (t = 2, 3, \ldots, m)$$

into a multiplication $x_t' = \lambda_t x_t$ $(t = 2, 3, \ldots, m)$.

Therefore $Q^{-1}(P^{-1}AP)Q$ is the multiplication M

$$x_1' = \lambda_1 x_1, \ x_2' = \lambda_2 x_2, \ \ldots, \ x_m' = \lambda_m x_m,$$

and A is transformed into M by the substitution $S = PQ$, which is unitary since P and Q are unitary.

Since $S^{-1}AS = M$, while A is Hermitian and S unitary, M is Hermitian. Therefore $\lambda_1, \lambda_2, \ldots, \lambda_m$ are real. Hence

All the characteristic-roots of a Hermitian substitution are real.

A *real symmetric* substitution A is a particular case of Hermitian substitution, and therefore its characteristic-roots are all real. It follows at once from the definition of a 'pole'

* It is also evidently Hermitian, and therefore of order 2.

(§ 6) that if $(X_1, X_2, \ldots, X_m)$ is a pole of A, $X_1, X_2, \ldots, X_m$ may be considered real. In this case P will be real and orthogonal (since a real unitary substitution is orthogonal by § 10). In fact the matrix of P becomes

$$\begin{vmatrix} X_1 & X_2 & X_3 & \cdot\;\cdot\;\cdot & X_m \\ X_2 & \dfrac{X_2^2}{X_1+1}-1 & \dfrac{X_2X_3}{X_1+1} & \cdot\;\cdot\;\cdot & \dfrac{X_2X_m}{X_1+1} \\ X_3 & \dfrac{X_3X_2}{X_1+1} & \dfrac{X_3^2}{X_1+1}-1 & \cdot\;\cdot\;\cdot & \dfrac{X_3X_m}{X_1+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ X_m & \dfrac{X_mX_2}{X_1+1} & \dfrac{X_mX_3}{X_1+1} & \cdot\;\cdot\;\cdot & \dfrac{X_m^2}{X_1+1}-1 \end{vmatrix} \quad \ldots\text{(ii)}$$

where $$X_1^2+X_2^2+\ldots+X_m^2=1.$$

Hence

A real symmetric substitution has all its characteristic-roots real, and may be transformed by a real orthogonal substitution into a real multiplication.

Again, suppose A is a *real alternate* substitution. Its degree must be even, if its determinant does not vanish; for a skew-symmetric determinant of odd order vanishes.

Then TA is Hermitian, where T is the similarity substitution

$$x_1'=ix_1,\ x_2'=ix_2,\ \ldots,\ x_m'=ix_m.$$

We can find then a unitary substitution P such that $P^{-1}.TA.P=M$, where M is a real multiplication. This gives $P^{-1}AP=P^{-1}T^{-1}P.M$. But $P^{-1}T^{-1}P=T^{-1}$; so that $P^{-1}AP=T^{-1}M$, which is a multiplication, whose coefficients are pure imaginaries.

Hence

A real alternate substitution has all its characteristic-roots pure imaginaries, and may be transformed by a unitary substitution into a multiplication whose coefficients are pure imaginaries.

Ex. 1. The substitution A with matrix

$$\begin{vmatrix} \frac{317}{225} & -\frac{106}{225} & \frac{80}{225} \\ -\frac{106}{225} & -\frac{142}{225} & -\frac{190}{225} \\ \frac{80}{225} & -\frac{190}{225} & \frac{275}{225} \end{vmatrix}$$

has a pole $(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3})$ corresponding to the characteristic-root 2.

It is transformed by the orthogonal substitution P with matrix

$$\begin{vmatrix} -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

into the substitution with matrix

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & -\frac{7}{25} & -\frac{24}{25} \\ 0 & -\frac{24}{25} & \frac{7}{25} \end{vmatrix}.$$

This has a pole $(0, \frac{3}{5}, -\frac{4}{5})$ corresponding to the characteristic-root 1, and is transformed by $Q \equiv (x, \frac{3}{5}y-\frac{4}{5}z, -\frac{4}{5}y-\frac{3}{5}z)$ into $(2x, y, -z)$.

Therefore A is transformed by the orthogonal substitution PQ with matrix

$$\begin{vmatrix} -\frac{10}{15} & \frac{5}{15} & -\frac{10}{15} \\ \frac{11}{15} & \frac{2}{15} & -\frac{10}{15} \\ \frac{2}{15} & \frac{14}{15} & \frac{5}{15} \end{vmatrix}$$

into $(2x, y, -z)$.

Ex. 2. Transform into a multiplication the symmetric substitution with matrix

$$\begin{vmatrix} -\frac{1}{3} & -\frac{4}{3} & \frac{2}{3} \\ -\frac{4}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{vmatrix}.$$

Ex. 3. Given a substitution B, find a unitary substitution A so that $C\bar{C}'$ is a real multiplication, where $C = AB$.

[A^{-1} is the unitary substitution transforming the Hermitian substitution $B\bar{B}'$ into a real multiplication.]

Ex. 4. A skew-Hermitian substitution A for which $a_{ij} = -\bar{a}_{ji}$ has all its characteristic-roots pure imaginaries, and can be transformed into a multiplication.

[SA is Hermitian, where $S \equiv (ix_1, ix_2, ix_3, \ldots)$.]

§ 13.

The result that a Hermitian substitution (and in particular a real symmetric substitution) has all its characteristic-roots real is so important that we give another proof. The result is evidently a particular case of the following theorem :—

The equation

$$|a-\lambda b| \equiv \begin{vmatrix} a_{11}-\lambda b_{11}, & a_{12}-\lambda b_{12}, & \ldots, & a_{1m}-\lambda b_{1m} \\ a_{21}-\lambda b_{21}, & a_{22}-\lambda b_{22}, & \ldots, & a_{2m}-\lambda b_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1}-\lambda b_{m1}, & a_{m2}-\lambda b_{m2}, & \ldots, & a_{mm}-\lambda b_{mm} \end{vmatrix} = 0$$

has all its roots real, if $a_{ij} = \bar{a}_{ji}$, $b_{ij} = \bar{b}_{ji}$, *and one or other of the Hermitian forms* $\sum_{i,j} a_{ij}\bar{x}_i x_j$, $\sum_{i,j} b_{ij}\bar{x}_i x_j$* *is* definite, *i.e. does not vanish for any values of* $x_1, x_2, \ldots, x_m$ *not all zero.*

If we take $\sum_{i,j} b_{ij}\bar{x}_i x_j \equiv x_1\bar{x}_1 + x_2\bar{x}_2 + \ldots + x_m\bar{x}_m$, we have the fact that the characteristic-roots of the Hermitian substitution A are all real.

Choose $X_1, X_2, \ldots, X_m$ to satisfy

$$a_{t1}X_1 + a_{t2}X_2 + \ldots + a_{tm}X_m = \lambda\,(b_{t1}X_1 + b_{t2}X_2 + \ldots + b_{tm}X_m)\ (t = 1, 2, \ldots, m).$$

This will be possible if $|a - \lambda b| = 0$.

Then $\sum_{i,j} a_{ij}\bar{X}_i X_j = \sum_i \bar{X}_i\,(a_{i1}X_1 + a_{i2}X_2 + \ldots + a_{im}X_m)$

$$= \lambda \sum_i \bar{X}_i\,(b_{i1}X_1 + b_{i2}X_2 + \ldots + b_{im}X_m) = \lambda \sum_{i,j} b_{ij}\bar{X}_i X_j.$$

Now since $a_{ij}\bar{X}_i X_j$ and $a_{ji}\bar{X}_j X_i$ are conjugate complex quantities, $\sum_{i,j} a_{ij}\bar{X}_i X_j$ and similarly $\sum_{i,j} b_{ij}\bar{X}_i X_j$ are real.

Hence λ is real, since by hypothesis $\Sigma a_{ij}\bar{X}_i X_j$ and $\Sigma b_{ij}\bar{X}_i X_j$ are not both zero.

If a_{ij} is always *real*, $\sum_{i,j} a_{ij}\bar{x}_i x_j$ will be definite whenever the real quadratic form $\sum_{i,j} a_{ij}x_i x_j$ is definite, i.e. does not vanish for any *real* values of $x_1, x_2, \ldots, x_m$ not all zero. For if $x_t = \xi_t + i\eta_t$, $\Sigma a_{ij}\bar{x}_i x_j = \Sigma a_{ij}\xi_i\xi_j + \Sigma a_{ij}\eta_i\eta_j$; ξ_t, η_t, a_{ij} being real.

Similarly if b_{ij} is always real.

Ex. 1. Show that $|a - \lambda b| = 0$ has not necessarily real roots if neither $\Sigma a_{ij}\bar{x}_i x_j$ nor $\Sigma b_{ij}\bar{x}_i x_j$ is definite.

[For instance, consider $\begin{vmatrix} \lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$; for which

$$X_1 = 1,\ X_2 = -i.]$$

Ex. 2. If both $\Sigma a_{ij}\bar{x}_i x_j$ and $\Sigma b_{ij}\bar{x}_i x_j$ are definite, the roots of $|a - \lambda b| = 0$ have all the same sign.

Ex. 3. Deduce from § 13, that

$$\begin{vmatrix} x & a & b & c & \cdot & \cdot & \cdot \\ -a & x & d & e & \cdot & \cdot & \cdot \\ -b & -d & x & f & \cdot & \cdot & \cdot \\ -c & -e & -f & x & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0$$

has purely imaginary roots, if $a, b, c, d, e, f, \ldots$ are real.

[Multiply every row by i.]

* See footnote, p. 34.

Ex. 4. Through the curve of intersection of an ellipsoid S with another conicoid S' three *real* paraboloids can be drawn.

[Let the cones joining the origin to the infinitely distant points on S and S' be

$$F \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0,$$

and $$F' \equiv a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy = 0.$$

Then $S-\lambda S'=0$ is a paraboloid if $F-\lambda F'=0$ is a pair of planes, i. e.

$$\begin{vmatrix} a-\lambda a' & h-\lambda h' & g-\lambda g' \\ h-\lambda h' & b-\lambda b' & f-\lambda f' \\ g-\lambda g' & f-\lambda f' & c-\lambda c' \end{vmatrix} = 0.$$

The roots of this equation are real if F is always positive.]

Ex. 5. An ellipsoid and a concentric conicoid have a real trio of common conjugate diameters.

§ 14. Transformation of a Unitary Substitution into a Multiplication.

A unitary substitution A can be transformed by a unitary substitution into a multiplication.

Let $(X_1, X_2, \ldots, X_m)$ be a pole of A as in § 12. Then if P is the substitution with matrix (i) of § 12, $P^{-1}AP = B$ is unitary, since the transform of a unitary substitution by a unitary substitution is unitary (§ 10).

Now, as in § 12, $b_{21} = b_{31} = \ldots = b_{m1} = 0$.

Therefore, since B is unitary, so that

$$b_{11}\bar{b}_{1t}+b_{21}\bar{b}_{2t}+\ldots+b_{m1}\bar{b}_{mt} = 0 \quad (t = 2, 3, \ldots, m),$$

we have $$b_{12} = b_{13} = \ldots = b_{1m} = 0.$$

The argument now goes on as in § 12.

Corollary.

The characteristic-roots of a unitary substitution A have unit modulus.

For the multiplication into which A is transformed by a unitary substitution is unitary; and the m coefficients of the multiplication are the characteristic-roots of A.

Ex. Transform the unitary substitution

$$x' = (i\cos^2\phi - \sin^2\phi)x + (1+i)\sin\phi \, . \cos\phi \, . y,$$
$$y' = (1+i)\sin\phi \, . \cos\phi \, . x + (i\sin^2\phi - \cos^2\phi)y$$

into a multiplication.

[The substitution has a pole $(\sin\phi, -\cos\phi)$ corresponding to the characteristic-root -1. Therefore the substitution

$$(\sin\phi \, . x - \cos\phi \, . y, \; -\cos\phi \, . x - \sin\phi \, . y)$$

transforms it into a multiplication $(-x, iy)$.]

§ 15. Canonical form of a Real Orthogonal Substitution.

A real orthogonal substitution A can be transformed by a real orthogonal substitution into the direct product of substitutions of the three types

$$\text{(i) } x' = x, \qquad \text{(ii) } x' = -x,$$
$$\text{(iii) } \left.\begin{aligned} x &= \cos\theta . x - \sin\theta . y \\ y &= \sin\theta . x + \cos\theta . y \end{aligned}\right\}.$$

Suppose A is

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \ (t = 1, 2, \ldots, m),$$

and that $(X_1, X_2, \ldots, X_m)$ is a pole corresponding to the characteristic-root λ.

Then

$$\lambda^2(X_1^2 + X_2^2 + \ldots + X_m^2) = (a_{11}X_1 + a_{12}X_2 + \ldots + a_{1m}X_m)^2 + \ldots$$
$$\ldots + (a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{mm}X_m)^2 = (X_1^2 + X_2^2 + \ldots + X_m^2),$$

since A is orthogonal.

Hence either $X_1^2 + X_2^2 + \ldots + X_m^2 = 0$ or $\lambda^2 = 1$.

The above theorem is true when $m = 1$. Suppose it true for all values of m less than the one considered.

(1) Let A have a pole $(X_1, X_2, \ldots, X_m)$ for which

$$X_1^2 + X_2^2 + \ldots + X_m^2 \neq 0.$$

Then $\lambda^2 = 1$, and from the definition of a pole in § 6 we may suppose $X_1, X_2, \ldots, X_m$ all real and such that

$$X_1^2 + X_2^2 + \ldots + X_m^2 = 1.$$

Take P as the orthogonal substitution with matrix (ii) of § 12. Then $P^{-1}AP = B$, where $b_{21} = b_{31} = \ldots = b_{m1} = 0$, as in § 12.

Since P and A are real and orthogonal, so is B.

Hence

$$b_{11}^2 + b_{21}^2 + \ldots + b_{m1}^2 = 1 \text{ and } b_{11}b_{1t} + b_{21}b_{2t} + \ldots + b_{m1}b_{mt} = 0$$
$$(t = 2, 3, \ldots, m);$$

so that $b_{11} = \pm 1$, $b_{12} = b_{13} = \ldots = b_{1m} = 0$.

Therefore A has been transformed by the real orthogonal substitution P into the direct product of $x_1' = \pm x_1$ and of

$$x_t' = b_{t2}x_2 + b_{t3}x_3 + \ldots + b_{tm}x_m \ (t = 2, 3, \ldots, m).$$

But by hypothesis the theorem is true for this last substitution of degree $(m-1)$.

Hence by induction as in § 12 the theorem is true for A.

(2) Let A have no pole $(X_1, X_2, \ldots, X_m)$ for which

$$X_1^2 + X_2^2 + \ldots + X_m^2 \neq 0.$$

Then it is impossible for $X_1, X_2, \ldots, X_m$ to be all real. We may suppose $X_1\bar{X}_1 + X_2\bar{X}_2 + \ldots + X_m\bar{X}_m = 2$.

Let $2p_{t1} = X_t + \bar{X}_t$, $2ip_{t2} = X_t - \bar{X}_t$ $(t = 1, 2, \ldots, m)$.

Then $p_{11}^2+p_{21}^2+\ldots+p_{m1}^2=p_{12}^2+p_{22}^2+\ldots+p_{m2}^2=1,$

$$p_{11}p_{12}+p_{21}p_{22}+\ldots+p_{m1}p_{m2}=0.$$

Choose real quantities $p_{13}, p_{23}, \ldots, p_{m3}$ to satisfy

$$p_{13}^2+p_{23}^2+\ldots+p_{m3}^2=1,$$

$$p_{11}p_{13}+p_{21}p_{23}+\ldots+p_{m1}p_{m3}=p_{12}p_{13}+p_{22}p_{23}+\ldots+p_{m2}p_{m3}=0.$$

Then choose real quantities $p_{14}, p_{24}, \ldots, p_{m4}$ to satisfy

$$p_{14}^2+p_{24}^2+\ldots+p_{m4}^2=1,$$

$$\begin{aligned}p_{11}p_{14}+p_{21}p_{24}+\ldots+p_{m1}p_{m4}&=p_{12}p_{14}+p_{22}p_{24}+\ldots+p_{m2}p_{m4}\\&=p_{13}p_{14}+p_{23}p_{24}+\ldots+p_{m3}p_{m4}=0,\end{aligned}$$

and so on.

The determinant $\begin{vmatrix} p_{11} & \cdot & \cdot & \cdot & p_{1m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{m1} & \cdot & \cdot & \cdot & p_{mm} \end{vmatrix}$

of the real orthogonal substitution P

$$x_t'=p_{t1}x_1+p_{t2}x_2+\ldots+p_{tm}x_m \quad (t=1, 2, \ldots, m)$$

obtained in this way will not vanish, for its square is unity.

Let $PAP^{-1}=D$, where D is the real orthogonal substitution $x_t'=d_{t1}x_1+d_{t2}x_2+\ldots+d_{tm}x_m$ $(t=1, 2, \ldots, m)$.

Then by § 6 (ii) D has $(1, \pm i, 0, 0, \ldots, 0)$ as poles corresponding to the poles $(X_1, X_2, \ldots, X_m)$ and $(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_m)$ of A, and hence

$$d_{31}=d_{41}=\ldots=d_{m1}=0, \quad d_{32}=d_{42}=\ldots=d_{m2}=0. \quad \ldots\ldots\text{(i)}$$

But, since D is orthogonal,

$$d_{11}^2+d_{21}^2+\ldots+d_{m1}^2=d_{12}^2+d_{22}^2+\ldots+d_{m2}^2=1,$$

$$d_{11}d_{12}+d_{21}d_{22}+\ldots+d_{m1}d_{m2}=0,$$

$$d_{1t}d_{11}+d_{2t}d_{21}+\ldots+d_{mt}d_{m1}=d_{1t}d_{12}+d_{2t}d_{22}+\ldots+d_{mt}d_{m2}=0$$
$$(t=3, 4, \ldots, m).$$

By (i) these reduce to

$$d_{11}^2+d_{21}^2=d_{12}^2+d_{22}^2=1, \quad d_{11}d_{12}+d_{21}d_{22}=0,$$

$$d_{1t}d_{11}+d_{2t}d_{21}=d_{1t}d_{12}+d_{2t}d_{22}=0;$$

which give readily

$$d_{13}=d_{14}=\ldots=d_{1m}=0, \quad d_{23}=d_{24}=\ldots=d_{2m}=0,$$

$$d_{11}=\cos\theta, \; d_{12}=-\sin\theta,$$

and either

$$d_{21}=\sin\theta, \; d_{22}=\cos\theta \quad \text{or} \quad d_{21}=-\sin\theta, \; d_{22}=-\cos\theta.$$

But the latter alternative is impossible, for then PAP^{-1} would have a pole $(Y_1, Y_2, \ldots, Y_m)$, where

$$Y_1 = \cos\frac{\theta}{2},\ Y_2 = -\sin\frac{\theta}{2},\ Y_3 = Y_4 = \ldots = Y_m = 0;$$

so that $$Y_1^2 + Y_2^2 + \ldots + Y_m^2 = 1.$$

Now by § 6 (ii)

$$X_t = p_{t1}Y_1 + p_{t2}Y_2 + \ldots + p_{tm}Y_m\ (t = 1, 2, \ldots, m),$$

and, squaring and adding these m equations, we get

$$Y_1^2 + Y_2^2 + \ldots + Y_m^2 = X_1^2 + X_2^2 + \ldots + X_m^2,$$

which is zero.

In fact

$$x_1' = \cos\theta\,.\,x_1 - \sin\theta\,.\,x_2,\ x_2' = -\sin\theta\,.\,x_1 - \cos\theta\,.\,x_2$$

is transformed by the orthogonal substitution

$$x_1' = \cos\frac{\theta}{2}\,.\,x_1 - \sin\frac{\theta}{2}\,.\,x_2,\ x_2' = \sin\frac{\theta}{2}\,.\,x_1 + \cos\frac{\theta}{2}\,.\,x_2$$

into $$x_1' = x_1,\ x_2' = -x_2.$$

Hence A has been transformed into the direct product of the real orthogonal substitutions

$$x_1' = \cos\theta\,.\,x_1 - \sin\theta\,.\,x_2,\ x_2' = \sin\theta\,.\,x_1 + \cos\theta\,.\,x_2,$$

and $$x_t' = d_{t3}x_3 + d_{t4}x_4 + \ldots + d_{tm}x_m\ (t = 3, 4, \ldots, m).$$

The proof by induction is now completed as in (1).

Corollary.

A real orthogonal substitution is transformable into a multiplication.

It is sufficient to show that

$$x_1' = \cos\theta\,.\,x_1 - \sin\theta\,.\,x_2,\ x_2' = \sin\theta\,.\,x_1 + \cos\theta\,.\,x_2$$

is transformable into a multiplication; which is the case by § 9, Corollary I, since it has the two distinct poles $(1, \pm i)$.

Ex. 1. Transform $x' = y,\ y' = z,\ z' = x$ into canonical form.

[A pole corresponding to the characteristic-root 1 is

$$\left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}\right).$$

Hence if P has the matrix

$$\begin{vmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{-\sqrt{3}-1}{2\sqrt{3}} & \frac{\sqrt{3}-1}{2\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{3}-1}{2\sqrt{3}} & \frac{-\sqrt{3}-1}{2\sqrt{3}} \end{vmatrix},$$

$P^{-1}AP$ is canonical.

In fact $P^{-1}AP$ is $\left(x,\ -\frac{1}{2}y + \frac{\sqrt{3}}{2}z,\ -\frac{\sqrt{3}}{2}y - \frac{1}{2}z\right)$].

Ex. 2. Transform similarly $x' = -y,\ y' = z,\ z' = x$.

CHAPTER II

INVARIANT-FACTORS

§ 1. Rank of a Determinant.

SUPPOSE that in the determinant *

$$\begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mm} \end{vmatrix}$$

of the substitution A the determinant itself and all the 1st, 2nd, ..., $(m-r-1)$-th minors vanish, but that not all the $(m-r)$-th minors vanish. Then the determinant is said to be of *rank* r. If the determinant does not vanish, it is of rank m; if the determinant vanishes but not every first minor vanishes, the determinant is of rank $m-1$; and so on. If every element a_{ij} vanishes, the determinant is of rank zero.

If B is a substitution whose determinant does not vanish, the determinants of AB and BA will have the same rank as the determinant of A.

For each k-th minor of the determinant of AB (or BA) is a linear function of k-th minors of the determinant of A; for instance, taking $k=3$ and $m=4$,

$$\begin{vmatrix} b_{11}a_{11}+b_{12}a_{21}+b_{13}a_{31}+b_{14}a_{41}, & b_{11}a_{12}+b_{12}a_{22}+b_{13}a_{32}+b_{14}a_{42}, & b_{11}a_{13}+b_{12}a_{23}+b_{13}a_{33}+b_{14}a_{43} \\ b_{21}a_{11}+b_{22}a_{21}+b_{23}a_{31}+b_{24}a_{41}, & b_{21}a_{12}+b_{22}a_{22}+b_{23}a_{32}+b_{24}a_{42}, & b_{21}a_{13}+b_{22}a_{23}+b_{23}a_{33}+b_{24}a_{43} \\ b_{31}a_{11}+b_{32}a_{21}+b_{33}a_{31}+b_{34}a_{41}, & b_{31}a_{12}+b_{32}a_{22}+b_{33}a_{32}+b_{34}a_{42}, & b_{31}a_{13}+b_{32}a_{23}+b_{33}a_{33}+b_{34}a_{43} \end{vmatrix}$$

* In §§ 1 to 4 we do not assume that the determinant of a substitution does not vanish.

$$= \begin{vmatrix} b_{12} b_{13} b_{14} \\ b_{22} b_{23} b_{24} \\ b_{32} b_{33} b_{34} \end{vmatrix} \times \begin{vmatrix} a_{21} a_{31} a_{41} \\ a_{22} a_{32} a_{42} \\ a_{23} a_{33} a_{43} \end{vmatrix} + \begin{vmatrix} b_{11} b_{13} b_{14} \\ b_{21} b_{23} b_{24} \\ b_{31} b_{33} b_{34} \end{vmatrix} \times \begin{vmatrix} a_{11} a_{31} a_{41} \\ a_{12} a_{32} a_{42} \\ a_{13} a_{33} a_{43} \end{vmatrix}$$

$$+ \begin{vmatrix} b_{11} b_{12} b_{14} \\ b_{21} b_{22} b_{24} \\ b_{31} b_{32} b_{34} \end{vmatrix} \times \begin{vmatrix} a_{11} a_{21} a_{41} \\ a_{12} a_{22} a_{42} \\ a_{13} a_{23} a_{43} \end{vmatrix} + \begin{vmatrix} b_{11} b_{12} b_{13} \\ b_{21} b_{22} b_{23} \\ b_{31} b_{32} b_{33} \end{vmatrix} \times \begin{vmatrix} a_{11} a_{21} a_{31} \\ a_{12} a_{22} a_{32} \\ a_{13} a_{23} a_{33} \end{vmatrix},$$

as is readily verified.

It follows immediately that the rank of the determinant of AB (or BA) cannot exceed the rank of the determinant of A.

But since $A = AB \, . \, B^{-1}$, the preceding argument shows that the rank of the determinant of A cannot exceed the rank of the determinant of AB.

Hence the ranks of the determinants of A and AB (or BA) are equal.

Ex. 1. The rank of a determinant is the same as the rank of the transposed determinant.

Ex. 2. The rank of a determinant is not altered by multiplying all the elements of any row by a non-zero constant, or by adding to each element of any row the corresponding element of another row.

Ex. 3. Find the rank of the determinants

$$\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}, \quad \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 2 & 1 & 0 \end{vmatrix},$$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

[2, 1, 1, 2, 1, 3.]

Ex. 4. Prove that the determinants of A and $A\bar{A}'$ have the same rank.

[Use the expression given in § 1 for a minor of the determinant of AB in terms of the minors of the determinants of A and B.]

Ex. 5. If the determinant of A is of rank r, the characteristic-equation of A has at least $m-r$ zero roots.

[Differentiate the characteristic-equation $m-r-1$ times with respect to λ. The resulting equation has a zero root.]

§ 2. Invariant-factors of a Determinant.

Consider now the determinant

$$\Delta \equiv \begin{vmatrix} a_{11}-\lambda b_{11} & a_{12}-\lambda b_{12} & \cdot & \cdot & \cdot & a_{1m}-\lambda b_{1m} \\ a_{21}-\lambda b_{21} & a_{22}-\lambda b_{22} & \cdot & \cdot & \cdot & a_{2m}-\lambda b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1}-\lambda b_{m1} & a_{m2}-\lambda b_{m2} & \cdot & \cdot & \cdot & a_{mm}-\lambda b_{mm} \end{vmatrix}.$$

When expanded it may be expressed in the form

$$k(\lambda-\alpha)^r(\lambda-\beta)^s(\lambda-\gamma)^t \ldots,$$

where k is independent of λ.

Suppose that *every* first minor when expressed in the same form has $(\lambda-\alpha)^{r_1}$ as a factor (but that one or more is not divisible by $(\lambda-\alpha)^{r_1+1}$), that *every* second minor has $(\lambda-\alpha)^{r_2}$ as a factor, and so on.*

Now $r > r_1$. For by the ordinary rule for differentiating a determinant † $\dfrac{\partial\Delta}{\partial\lambda}$ is a linear function of the first minors of Δ, and $\dfrac{\partial\Delta}{\partial\lambda}$ is divisible by $(\lambda-\alpha)^{r-1}$ but not by $(\lambda-\alpha)^r$.

Similarly $\qquad r_1 > r_2 > r_3 > \ldots.$

Suppose quantities $(\lambda-\beta)^{s_1}$, $(\lambda-\beta)^{s_2}$, $(\lambda-\beta)^{s_3}$, ..., &c., obtained in a similar manner.

Then the quantities

$$(\lambda-\alpha)^{r-r_1},\ (\lambda-\alpha)^{r_1-r_2},\ (\lambda-\alpha)^{r_2-r_3},\ \ldots;$$
$$(\lambda-\beta)^{s-s_1},\ (\lambda-\beta)^{s_1-s_2},\ (\lambda-\beta)^{s_2-s_3},\ \ldots;\ \&c., \&c.,$$

are called the *invariant-factors* ‡ of Δ.

The name is derived from the following important property:

Suppose P, Q any substitutions, and let $PAQ = C$, $PBQ = D$. Then, if

$$\delta \equiv \begin{vmatrix} c_{11}-\lambda d_{11} & c_{12}-\lambda d_{12} & \cdot & \cdot & \cdot & c_{1m}-\lambda d_{1m} \\ c_{21}-\lambda d_{21} & c_{22}-\lambda d_{22} & \cdot & \cdot & \cdot & c_{2m}-\lambda d_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{m1}-\lambda d_{m1} & c_{m2}-\lambda d_{m2} & \cdot & \cdot & \cdot & c_{mm}-\lambda d_{mm} \end{vmatrix},$$

Δ and δ have the same invariant-factors.

* We suppose zero divisible by every power of $(\lambda-\alpha)$.

† If dashes denote differentiation with respect to x,

$$\frac{d}{dx}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ g' & h' & i' \end{vmatrix},$$

and so in general.

‡ *Elementartheiler.*

Suppose that the highest power of $(\lambda-\alpha)$ which divides all the k-th minors of δ is $(\lambda-\alpha)^{\rho_k}$.

We assume for the present that any k-th minor of δ is a linear function of k-th minors of Δ.

It will follow that every k-th minor of δ is divisible by $(\lambda-\alpha)^{r_k}$, or $\rho_k \geq r_k$.

Similarly, since $A = P^{-1}CQ^{-1}$ and $B = P^{-1}DQ^{-1}$, we have $r_k \geq \rho_k$.

Hence $r_k = \rho_k$; and the identity of the invariant-factors of Δ and δ immediately follows.

In particular, δ/Δ is independent of λ. We readily show in fact that δ/Δ is the product of the determinants of P and Q.

To prove that any k-th minor of δ is a linear function of k-th minors of Δ, we proceed as follows:—

Let $AQ = U$ and $BQ = V$. Then, denoting by $A-\lambda B$ the substitution whose coefficients are $a_{ij}-\lambda b_{ij}$, we show exactly as in § 1 that any k-th minor of the determinant of $U-\lambda V$, i.e. of $(A-\lambda B)Q$, is a linear function of k-th minors of Δ.

Repeating the process, we show that any k-th minor of δ, i.e. of the determinant of $P(U-\lambda V)$, is a linear function of k-th minors of the determinant of $U-\lambda V$, and is therefore a linear function of k-th minors of Δ.

In the above we may suppose the determinant of A to be zero, if we admit zero values for $\alpha, \beta, \gamma, \ldots$.

If the determinant of B vanishes we may consider similarly the determinant whose elements are $\lambda a_{ij}-b_{ij}$.

It is essential, however, that the determinants of P and Q should not vanish.

We have supposed, moreover, that Δ does not vanish identically for every value of λ. For a discussion of this case we must refer elsewhere.*

Ex. 1. Find the invariant-factors of

$$\begin{vmatrix} 1-\lambda & 0 \\ 1-\lambda & 1-\lambda \end{vmatrix}, \quad \begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix}, \quad \begin{vmatrix} 2+\lambda, & \lambda \\ 1+2\lambda, & 2-\lambda \end{vmatrix},$$

$$\begin{vmatrix} 6-3\lambda & 2+\lambda & 3\lambda \\ 2+\lambda & -1+\lambda & 3 \\ 3\lambda & 3 & 2+\lambda \end{vmatrix}, \quad \begin{vmatrix} 1+\lambda & 0 & -1-2\lambda \\ 0 & 1+3\lambda & -1-4\lambda \\ -1-2\lambda & -1-4\lambda & 4\lambda \end{vmatrix},$$

$$\begin{vmatrix} 1+\lambda & 0 & 1+\lambda \\ 0 & 0 & 1-\lambda \\ 1+\lambda & 1-\lambda & -1+3\lambda \end{vmatrix}, \quad \begin{vmatrix} 0 & 0 & -2+\lambda \\ 0 & 1+\lambda & 2\lambda \\ -2+\lambda & 2\lambda & -8+4\lambda \end{vmatrix}.$$

* e. g. Muth, *Elementartheiler*, § 8; Bromwich, *Quadratic Forms*, § 21.

[$(\lambda-1), (\lambda-1)$; $(\lambda-1)^2$; $(\lambda-1), (\lambda-\frac{4}{3})$; $(\lambda-1), (\lambda+2), (\lambda-\frac{37}{13})$; $(\lambda+\frac{1}{2})^3$; $(\lambda-1), (\lambda-1), (\lambda+1)$; $(\lambda-2)^2, (\lambda+1)$.]

Ex. 2. Find the invariant-factors of

$$\begin{vmatrix} \alpha-\lambda & 1 & 0 & 0 \\ 0 & \alpha-\lambda & 1 & 0 \\ 0 & 0 & \alpha-\lambda & 1 \\ 0 & 0 & 0 & \alpha-\lambda \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ a & b & c & d-\lambda \end{vmatrix}.$$

[$(\lambda-\alpha)^4$; the factors of $\lambda^4-a-b\lambda-c\lambda^2-d\lambda^3$.]

Ex. 3. The invariant-factors of the determinant Δ of § 2 are unaltered if we multiply any row (or column) of Δ by a non-zero constant independent of λ, or if we add each element of a given row to the corresponding element of another given row.

[This result will assist in the practical calculation of invariant-factors.]

Ex. 4. The complex invariant-factors of the determinant $|a-\lambda b|$ of Ch. I, § 13, occur in pairs of the type $(\lambda-\alpha)^r$, $(\lambda-\bar{\alpha})^r$.

[Since $a_{ij} = \bar{a}_{ji}$ and $b_{ij} = \bar{b}_{ji}$, the minors of this determinant can evidently be grouped into conjugate pairs. For instance,

$$\begin{vmatrix} a_{11}-\lambda b_{11} & a_{12}-\lambda b_{12} \\ a_{31}-\lambda b_{31} & a_{32}-\lambda b_{32} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_{11}-\lambda b_{11} & a_{13}-\lambda b_{13} \\ a_{21}-\lambda b_{21} & a_{23}-\lambda b_{23} \end{vmatrix}$$

are conjugate complex quantities, λ being considered real. Hence, if $(\lambda-\alpha)^r$ divides all minors of a given order, so does $(\lambda-\bar{\alpha})^r$; and conversely.]

§ 3. Invariant-factors of a Substitution.

By the *invariant-factors of a substitution A* we mean the invariant-factors of its characteristic-determinant.

In this case the substitution B of § 2 is replaced by the unit substitution E.

Taking $Q = S$, $P = S^{-1}$ in § 2, and noting that

$$D = S^{-1}ES = E,$$

we see that the invariant-factors of $S^{-1}AS$ and of A are identical, i.e.:—

The invariant-factors of a substitution A are identical with the invariant-factors of any substitution into which A can be transformed.

The following theorems are an immediate consequence of the definitions:—

A substitution has the same invariant-factors as its transposed substitution.

If the invariant-factors of A are

$$(\lambda-\alpha)^{a_1},\ (\lambda-\alpha)^{a_2},\ (\lambda-\alpha)^{a_3},\ \ldots,\ (\lambda-\beta)^{b_1},\ (\lambda-\beta)^{b_2},\ \ldots,\ \ldots,$$

the invariant-factors of $\bar{A}$ and $\bar{A}'$ are

$$(\lambda-\bar{\alpha})^{a_1},\ (\lambda-\bar{\alpha})^{a_2},\ (\lambda-\bar{\alpha})^{a_3},\ \ldots,\ (\lambda-\bar{\beta})^{b_1},\ (\lambda-\bar{\beta})^{b_2},\ \ldots,\ \ldots.$$

Again,

The invariant-factors of TA, where T is the similarity substitution

$$x_1' = kx_1,\ x_2' = kx_2,\ \ldots,\ x_m' = kx_m,$$

are

$$(\lambda-k\alpha)^{a_1},\ (\lambda-k\alpha)^{a_2},\ (\lambda-k\alpha)^{a_3},\ \ldots,\ (\lambda-k\beta)^{b_1},\ (\lambda-k\beta)^{b_2},\ \ldots.$$

Ex. 1. The complex invariant-factors of a *real* substitution occur in pairs of the type $(\lambda-\alpha)^a$, $(\lambda-\bar{\alpha})^a$.

Ex. 2. The complex invariant-factors of the product AM of a Hermitian substitution A and a real multiplication M occur in pairs of the type $(\lambda-\alpha)^a$, $(\lambda-\bar{\alpha})^a$.

[If $M \equiv (e_1x_1,\ e_2x_2,\ \ldots,\ e_mx_m)$, the characteristic-determinant of AM becomes after division by $e_1e_2\ldots e_m$

$$\begin{vmatrix} a_{11}-\lambda/e_1 & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22}-\lambda/e_2 & \cdots & a_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mm}-\lambda/e_m \end{vmatrix}.$$

Now use § 2, Ex. 4.]

Ex. 3. Prove similarly that the invariant-factors of the product of an alternate substitution and a multiplication occur in pairs of the type $(\lambda-\alpha)^a$, $(\lambda+\alpha)^a$.

§ 4. Invariant-factors of a Direct-Product.

It is easy to determine the invariant-factors of a direct product when we know the invariant-factors of the constituent substitutions.

Consider, for instance, a direct product of substitutions on the variables

$$x_1,\ x_2,\ x_3,\ \text{and}\ x_4,\ x_5,\ x_6,\ \text{and}\ x_7,\ x_8.$$

The characteristic-determinant of the substitution takes the form

$$\begin{vmatrix} *-\lambda & * & * & o & o & o & o & o \\ * & *-\lambda & * & o & o & o & o & o \\ * & * & *-\lambda & o & o & o & o & o \\ o & o & o & *-\lambda & * & * & o & o \\ o & o & o & * & *-\lambda & * & o & o \\ o & o & o & * & * & *-\lambda & o & o \\ o & o & o & o & o & o & *-\lambda & * \\ o & o & o & o & o & o & * & *-\lambda \end{vmatrix},$$

where the asterisks denote quantities not necessarily zero.

If we erase a row and a column of this determinant, the resulting first minor is readily seen to vanish unless the erased row is one of the first three rows and also the erased column one of the first three columns, or else the erased row is one of the next three rows and the erased column one of the next three columns, or else the erased row is one of the last two rows and the erased column is one of the last two columns.

In fact the resulting first minor will vanish unless the erasing of row and column leaves each of the three parts of the characteristic-determinant which is of the type

$$\begin{bmatrix} *-\lambda & * & * \\ * & *-\lambda & * \\ * & * & *-\lambda \end{bmatrix}$$

in square form; i.e. does not convert it into a rectangle with length and breadth unequal.

Similarly, when two rows and columns are erased the resulting second minor vanishes unless each part of the characteristic-determinant remains in square form; and so for third, fourth, ..., minors.

It follows at once that if each constituent has a *single* invariant-factor which is a power of $(\lambda-\alpha)$,* then the invariant-factors of the direct product corresponding to the characteristic-root α are

$$(\lambda-\alpha)^{a_1},\ (\lambda-\alpha)^{a_2},\ (\lambda-\alpha)^{a_3},\ \ldots,$$

where

$$(\lambda-\alpha)^{a_1},\ (\lambda-\alpha)^{a_2},\ (\lambda-\alpha)^{a_3},\ \ldots \qquad (a_1 \geq a_2 \geq a_3 \geq \ldots)$$

are the invariant-factors of the constituent substitutions.

Even when the constituent-substitutions have more than one invariant-factor which is a power of $(\lambda-\alpha)$, it will be easy to obtain the invariant-factors of the direct product in any given example; but the case just discussed is the case of most importance in the general theory.

§ 5. Invariant-factors of a Canonical Substitution.

The substitution C of Ch. I, § 8, has the single invariant-factor $(\lambda-\alpha)^m$. For the characteristic-determinant is $(\alpha-\lambda)^m$, and the first minor obtained by erasing the last row and the first column in this determinant is unity.

Using § 4, we at once obtain the invariant-factors of the *canonical* substitution of Ch. I, § 9, and establish that

* i.e. the determinant of each constituent has at least one first minor not divisible by $(\lambda-\alpha)$.

The direct product N of substitutions of the type
$x_1' = \mu x_1 + x_2,\ x_2' = \mu x_2 + x_3,\ \ldots,\ x'_{m-1} = \mu x_{m-1} + x_m,\ x_m' = \mu x_m,$
for which

$$\begin{array}{llll} \mu=\alpha \text{ and } m=a_1, & \mu=\alpha \text{ and } m=a_2, & \mu=\alpha \text{ and } m=a_3, & \ldots, \\ \mu=\beta \text{ and } m=b_1, & \mu=\beta \text{ and } m=b_2, & \mu=\beta \text{ and } m=b_3, & \ldots, \\ \mu=\gamma \text{ and } m=c_1, & \mu=\gamma \text{ and } m=c_2, & \mu=\gamma \text{ and } m=c_3, & \ldots, \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

respectively, where

$$a_1 \geq a_2 \geq a_3 \geq \ldots,\ b_1 \geq b_2 \geq b_3 \geq \ldots,\ c_1 \geq c_2 \geq c_3 \geq \ldots,\ \ldots,$$

has the invariant-factors

$$(\lambda-\alpha)^{a_1},\ (\lambda-\alpha)^{a_2},\ (\lambda-\alpha)^{a_3},\ \ldots,\ (\lambda-\beta)^{b_1},\ (\lambda-\beta)^{b_2},\ (\lambda-\beta)^{b_3},\ \ldots,$$
$$(\lambda-\gamma)^{c_1},\ (\lambda-\gamma)^{c_2},\ (\lambda-\gamma)^{c_3},\ \ldots,\ \ldots.$$

The reader will have no difficulty in showing by a similar process that the inverse N^{-1} of this direct product has the invariant-factors

$$\begin{array}{llll} (\lambda-\alpha^{-1})^{a_1}, & (\lambda-\alpha^{-1})^{a_2}, & (\lambda-\alpha^{-1})^{a_3}, & \ldots, \\ (\lambda-\beta^{-1})^{b_1}, & (\lambda-\beta^{-1})^{b_2}, & (\lambda-\beta^{-1})^{b_3}, & \ldots, \\ (\lambda-\gamma^{-1})^{c_1}, & (\lambda-\gamma^{-1})^{c_2}, & (\lambda-\gamma^{-1})^{c_3}, & \ldots,\ \ldots. \end{array}$$

Corollary I.

If a substitution A has invariant-factors

$$\begin{array}{llll} (\lambda-\alpha)^{a_1}, & (\lambda-\alpha)^{a_2}, & (\lambda-\alpha)^{a_3}, & \ldots, \textit{ where } a_1 \geq a_2 \geq a_3 \geq \ldots, \\ (\lambda-\beta)^{b_1}, & (\lambda-\beta)^{b_2}, & (\lambda-\beta)^{b_3}, & \ldots, \textit{ where } b_1 \geq b_2 \geq b_3 \geq \ldots, \\ (\lambda-\gamma)^{c_1}, & (\lambda-\gamma)^{c_2}, & (\lambda-\gamma)^{c_3}, & \ldots, \textit{ where } c_1 \geq c_2 \geq c_3 \geq \ldots, \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

it is transformable into the canonical substitution N of § 5.

For A is transformable into a canonical substitution (Ch. I, § 9), and any canonical substitution with the given invariant-factors is evidently transformable into N of § 5.

Corollary II.

If a substitution has the invariant-factors given in Corollary I, the invariant-factors of its inverse are

$$\begin{array}{llll} (\lambda-\alpha^{-1})^{a_1}, & (\lambda-\alpha^{-1})^{a_2}, & (\lambda-\alpha^{-1})^{a_3}, & \ldots, \\ (\lambda-\beta^{-1})^{b_1}, & (\lambda-\beta^{-1})^{b_2}, & (\lambda-\beta^{-1})^{b_3}, & \ldots, \\ (\lambda-\gamma^{-1})^{c_1}, & (\lambda-\gamma^{-1})^{c_2}, & (\lambda-\gamma^{-1})^{c_3}, & \ldots, \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

For if $S^{-1}AS = N$, $S^{-1}A^{-1}S = N^{-1}$ (Ch. I, § 5). Now use the result at the end of § 5.

Corollary III.

Any given substitution A can be transformed into any given substitution B with the same invariant-factors.

For by Corollary I we can find substitutions S, T such that $S^{-1}AS = N$, $T^{-1}BT = N$; and then $S^{-1}AS = T^{-1}BT$ or $P^{-1}AP = B$, where $P \equiv ST^{-1}$.

Equating corresponding coefficients in the two substitutions AP and PB of degree m, we have m^2 linear equations in the m^2 coefficients of P. If A and B have the same invariant-factors, these equations are consistent (though not in general independent). Solving them, we get a substitution P transforming A into B, whose coefficients are rational functions of the coefficients of A and B.

For instance, when we know the invariant-factors of A, we know N; and hence we can find a substitution transforming A into its canonical form N.*

Corollary IV.

A substitution A is of finite order if and only if every invariant-factor has unit exponent† and every characteristic-root is a root of unity.

For A is of the same order and has the same characteristic-roots and invariant-factors as the canonical form N into which it may be transformed. But N is only of finite order if it is a multiplication whose coefficients are roots of unity. (Ch. I, § 9, Corollary II.)

Corollary V.

If the substitution A has exactly k invariant-factors which are powers of $(\lambda-\alpha)$, it has a $(k-1)$-ply infinite number of poles corresponding to the characteristic-root $\lambda = \alpha$.

For this has been proved for N in Ch. I, §§ 7 and 8. Now use Ch. I, § 6.

Ex. 1. $(x+3y,\ 2x+2y)$ can be transformed into

$$(5x+6y,\ -x-2y).$$

[They both have invariant-factors $(\lambda+1)$, $(\lambda-4)$.]

* To find these invariant-factors, we may with advantage first transform A into some simpler form, e. g. by the process of Ch. IV, § 1, Ex. 10.

† An invariant-factor such as $(\lambda-\alpha)$ with unit exponent will often be called a *linear* or *simple* invariant-factor.

Ex. 2. $(3x+y-2z,\ 2x+3y-3z,\ 4x+2y-3z)$ is transformable into $(-z,\ x+3y+3z,\ -y)$.

[They both have the invariant-factor $(\lambda-1)^3$.]

Ex. 3. Find by invariant-factors the canonical form of

$$(x-y+2z-2w,\ -3y-2w,\ -2x+2y-3z+3w,\ 2y+w).$$

[See Ch. I, § 9, Ex. 3.]

Find similarly the canonical form of the substitutions in Ch. I, § 9, Ex. 5.

Ex. 4. Transform $A\equiv(x+14y+7z,\ -y-z,\ 6y+4z)$ into its canonical form $N\equiv(x,\ y,\ 2z)$.

[If $P^{-1}AP=N$, where

$$P\equiv(l_1x+m_1y+n_1z,\ l_2x+m_2y+n_2z,\ l_3x+m_3y+n_3z),$$

equating corresponding coefficients in $AP=PN$ gives

$$\left.\begin{array}{lll} l_1=l_1, & 14l_1-m_1+6n_1=m_1, & 7l_1-m_1+4n_1=n_1 \\ l_2=l_2, & 14l_2-m_2+6n_2=m_2, & 7l_2-m_2+4n_2=n_2 \\ l_3=2l_3, & 14l_3-m_3+6n_3=2m_3. & 7l_3-m_3+4n_3=2n_3 \end{array}\right\}.$$

These equations reduce to

$$7l_1=m_1-3n_1,\ 7l_2=m_2-3n_2,\ l_3=0,\ m_3=2n_3.$$

Hence P may be any substitution whose coefficients are connected by these four independent and consistent relations.]

Apply the method to Ex. 1, 2, 3.

§ 6. A Second Canonical Substitution.

The substitution N of § 5 is transformable into another form which is sometimes more useful than the form N itself.

Suppose that on page 57

$$(\lambda-\alpha)^{a_1}.(\lambda-\beta)^{b_1}.(\lambda-\gamma)^{c_1}\ldots\equiv\lambda^r-e_1-e_2\lambda-\ldots-e_r\lambda^{r-1},$$
$$(\lambda-\alpha)^{a_2}.(\lambda-\beta)^{b_2}.(\lambda-\gamma)^{c_2}\ldots\equiv\lambda^s-f_1-f_2\lambda-\ldots-f_s\lambda^{s-1},$$
$$(\lambda-\alpha)^{a_3}.(\lambda-\beta)^{b_3}.(\lambda-\gamma)^{c_3}\ldots\equiv\lambda^t-g_1-g_2\lambda-\ldots-g_t\lambda^{t-1},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

so that $a_1+b_1+c_1+\ldots=r,\ a_2+b_2+c_2+\ldots=s$, &c.

Then N is transformable into the substitution L

$$x_1'=x_2,\ x_2'=x_3,\ldots,\ x'_{r-1}=x_r,\ x_r'=e_1x_1+e_2x_2+\ldots+e_rx_r;$$
$$x'_{r+1}=x_{r+2},\ x'_{r+2}=x_{r+3},\ldots,\ x'_{r+s-1}=x_{r+s},$$
$$x'_{r+s}=f_1x_{r+1}+f_2x_{r+2}+\ldots+f_sx_{r+s};$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

which is the direct product of substitutions each of the type

$$x_1'=x_2,\ x_2'=x_3,\ \ldots,\ x'_{r-1}=x_r,\ x_r'=e_1x_1+e_2x_2+\ldots+e_rx_r,$$

the characteristic-determinant of each constituent of the direct product being a factor of the characteristic-determinant of the preceding constituent.

For since the first minor obtained by erasing the last row and the first column in the characteristic-determinant of

$$x_1' = x_2,\, x_2' = x_3,\, \ldots,\, x'_{r-1} = x_r,\, x_r' = e_1x_1 + e_2x_2 + \ldots + e_rx_r$$

is unity, this constituent of L has invariant-factors

$$(\lambda-\alpha)^{a_1},\ (\lambda-\beta)^{b_1},\ (\lambda-\gamma)^{c_1},\ \ldots.$$

It follows from § 4 that L has the same invariant-factors as N, and hence is transformable into N.*

The reader will find properties of the substitution L in Ch. I, § 3, Ex. 14, and § 6, Ex. 5 (iv).

If A is any substitution with canonical substitution N, the first constituent in the direct product L has a characteristic-determinant which is the quotient of the characteristic-determinant of A by the highest common factor of the first minors of the characteristic-determinant of A. The second constituent of L has a characteristic-determinant which is the quotient of this highest common factor by the highest common factor of the second minors of the characteristic-determinant of A; and so on. Hence L can always be found, when A is given; and the coefficients of L are rational functions of the coefficients of A.

Ex. 1. Transform

$$x_1' = \alpha x_1 + x_2,\ x_2' = \alpha x_2 + x_3,\ \ldots,\ x'_{r-1} = \alpha x_{r-1} + x_r,\ x_r' = \alpha x_r$$

into the type

$$x_1' = x_2,\ x_2' = x_3,\ \ldots,\ x'_{r-1} = x_r,\ x_r' = e_1x_1 + e_2x_2 + \ldots + e_rx_r.$$

[Put $x_1 = \xi_1,\ \alpha x_1 + x_2 = \xi_2,\ \alpha^2x_1 + 2\alpha x_2 + x_3 = \xi_3,\ \ldots,$

$$\alpha^k x_1 + {}^kC_1\alpha^{k-1}x_2 + {}^kC_2\alpha^{k-2}x_3 + \ldots + x_{k+1} = \xi_{k+1},\ \ldots.]$$

Ex. 2. If $A \equiv (x+14y+7z,\ -y-z,\ 6y+4z)$,

then $L \equiv (y,\ -2x+3y,\ z)$.

Find P so that $P^{-1}AP = L$.

[The invariant-factors of A are $(\lambda-1)$, $(\lambda-1)$, $(\lambda-2)$. This gives L.

If $P \equiv (l_1x + m_1y + n_1z,\ l_2x + m_2y + n_2z,\ l_3x + m_3y + n_3z)$,

we have, on comparing coefficients in AP and PL as in § 5,

* The following authors have made use of the substitution L: Nicoletti, *Annali di Matematica* III, xiv (1908); Landsberg, *Crelle*, cxvi (1896), p. 331; Burnside, *Proc. London Math. Soc.*, xxx (1898), p. 183; Lattès, *Comptes rendus*, clv (1912), p. 1482. For these references I am indebted to Professor Lattès.

Ex. 4, nine equations which reduce to the five independent and consistent equations

$$7l_3-m_3+3n_3=0,\ l_1=l_2,\ 7l_1-m_1+4n_1=n_2,$$
$$7l_2-m_2+2n_1+n_2=0,\ m_1-m_2=2n_1-2n_2.$$

For instance, we might have

$$P\equiv(x-2z,\ x+2y-z,\ -x-y+2z).]$$

Apply a similar process to § 5, Ex. 1, 2, 3.

Ex. 3. We can transform A into the form L by a substitution whose coefficients are rational functions of the coefficients of A.

[Use the method of Ex. 2.]

§ 7. Substitution Transformable into its Inverse.

As an example on the substitution of § 6 we may show that:—

If a substitution A has the same invariant-factors as its inverse, A is the product of two substitutions of order 2, each of which transforms A into its inverse.

If $P^{-1}AP=A^{-1}$ and $P^2=E$, then

$$(P^{-1}A)^{-1}.A\,(P^{-1}A)=A^{-1}.PAP^{-1}.A$$
$$=A^{-1}.P^{-1}AP.A=A^{-1}$$

and $\qquad (P^{-1}A)^2=P^{-1}A\,P^{-1}A=P^{-1}A\,P.A=E.$

Hence A is the product of two substitutions P and $P^{-1}A$ of order 2, each of which transforms A into A^{-1}.

It is therefore sufficient to prove the existence of P.

Moreover, it is sufficient to prove the existence of P for any substitution L into which A can be transformed; for if $P^{-1}LP=L^{-1}$ and $P^2=E$, where $S^{-1}AS=L$, then

$$(SPS^{-1})^{-1}A\,(SPS^{-1})=SP^{-1}.S^{-1}AS.PS^{-1}$$
$$=SP^{-1}LPS^{-1}=SL^{-1}S^{-1}=A^{-1}$$

and SPS^{-1} is of order 2.

We may take L as the substitution of § 6, which is the direct product of substitutions of the type

$$x_1'=x_2,\ x_2'=x_3,\ \ldots,\ x'_{r-1}=x_r,$$
$$x_r'=e_1x_1+e_2x_2+\ldots+e_rx_r\ \ldots\ldots\text{(i)}$$

It will evidently be sufficient to prove the result for this constituent (i). Now (i) is transformed by the substitution

$$x_1'=x_r,\ x_2'=x_{r-1},\ x_3'=x_{r-2},\ \ldots,\ x_r'=x_1$$

of order 2 into

$$x_1' = e_r x_1 + e_{r-1} x_2 + \dots + e_1 x_r,\ x_2' = x_1,$$
$$x_3' = x_2,\ \dots,\ x_r' = x_{r-1} \quad \text{......(ii)},$$

while the inverse of (i) is

$$x_1 = \frac{1}{e_1}(-e_2 x_1 - e_3 x_2 - \dots - e_r x_{r-1} + x_r),$$
$$x_2' = x_1,\ x_3' = x_2,\ \dots,\ x_r' = x_{r-1}.$$

This inverse coincides with (ii), provided

$$e_1 e_r = -e_2,\ e_1 e_{r-1} = -e_3,\ \dots,\ e_1 e_2 = -e_r,\ e_1^2 = 1 \quad \text{...... (iii)}$$

But these relations (iii) hold good. For since (i) is transformable into its inverse, and has therefore the same invariant-factors as its inverse, the characteristic-equation of (i)

$$\lambda^r - e_1 - e_2 \lambda - \dots - e_r \lambda^{r-1} = 0$$

must be identical with an expression of the type

$$(\lambda-1)^p (\lambda+1)^q (\lambda-\alpha)^a (\lambda-\alpha^{-1})^a (\lambda-\beta)^b (\lambda-\beta^{-1})^b \dots = 0$$

by Corollary II of § 5.

Hence $\lambda^r - e_1 - e_2\lambda - \dots - e_r\lambda^{r-1} = 0$ is unaltered when we replace λ by λ^{-1}; which gives the equations (iii).

CHAPTER III

BILINEAR FORMS

§ 1. Transformation of Bilinear Forms.

WITH any substitution A

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \dots + a_{tm}x_m \; (t = 1, 2, \dots, m)$$

may be associated a *bilinear form*

$$a(x, y) \equiv \Sigma a_{ij} y_i x_j \quad (i, j = 1, 2, \dots, m)$$

in the two sets of variables

$$x_1, x_2, \dots, x_m; \; y_1, y_2, \dots, y_m.$$

The product of two forms $a(x, y)$ and $b(x, y)$ is defined as $c(x, y)$, where

$$c_{ij} = b_{i1}a_{1j} + b_{i2}a_{2j} + \dots + b_{im}a_{mj}.$$

The substitution C corresponding to $c(x, y)$ is therefore the product of A and B corresponding to $a(x, y)$ and $b(x, y)$ respectively. It should be noticed that this definition will hold even if the determinant formed by the coefficients of $a(x, y)$ or $b(x, y)$ vanishes.

Suppose that in the form $a(x, y)$ we substitute

$$\left.\begin{array}{l} p_{t1}\mathbf{x}_1 + p_{t2}\mathbf{x}_2 + \dots + p_{tm}\mathbf{x}_m \text{ for } x_t \\ q_{t1}\mathbf{y}_1 + q_{t2}\mathbf{y}_2 + \dots + q_{tm}\mathbf{y}_m \text{ for } y_t \end{array}\right\} (t = 1, 2, \dots, m).$$

Then we transform $a(x, y) \equiv \sum_{i,j} a_{ij} y_i x_j$ into

$$\sum_{i,j} a_{ij}(q_{i1}\mathbf{y}_1 + q_{i2}\mathbf{y}_2 + \dots + q_{im}\mathbf{y}_m)(p_{j1}\mathbf{x}_1 + p_{j2}\mathbf{x}_2 + \dots + p_{jm}\mathbf{x}_m)$$

$$= \sum_{s,t}\Big(\sum_{i,j} q_{is}a_{ij}p_{jt}\Big)\mathbf{y}_s\mathbf{x}_t = \sum_{s,t} b_{st}\mathbf{y}_s\mathbf{x}_t = b(\mathbf{x}, \mathbf{y}),$$

where $B = PAQ'$; a result of considerable importance.

It will be noticed that by Ch. II, § 1, the determinants of B and A have the same rank.

If $P = Q$, the x's and y's are said to be transformed by a *congruent* transformation.

We have supposed the two sets of variables

$$x_1, x_2, \dots, x_m \text{ and } y_1, y_2, \dots, y_m$$

independent. This is not necessarily the case.

We might put $y_1 = x_1, y_2 = x_2, \ldots, y_m = x_m$.

Then $a(x, y)$ becomes the *quadratic form* $a(x, x)$ or

$$\Sigma a_{ij} x_i x_j \; (i, j = 1, 2, \ldots, m).$$

Any transformation of the x's and y's must now be congruent; so that if we put

$$p_{t1}\mathbf{x}_1 + p_{t2}\mathbf{x}_2 + \ldots + p_{tm}\mathbf{x}_m \text{ for } x_t$$

in $a(x, x)$ we get $b(\mathbf{x}, \mathbf{x})$, where $B = PAP'$.

Another case of importance is that in which

$$y_1 = \bar{x}_1, \; y_2 = \bar{x}_2, \; \ldots, \; y_m = \bar{x}_m.$$

Then $a(x, y)$ becomes $a(x, \bar{x})$ or $\Sigma a_{ij} \bar{x}_i x_j$.

If we put

$$p_{t1}\mathbf{x}_1 + p_{t2}\mathbf{x}_2 + \ldots + p_{tm}\mathbf{x}_m \text{ for } x_t,$$

we must put

$$\bar{p}_{t1}\bar{\mathbf{x}}_1 + \bar{p}_{t2}\bar{\mathbf{x}}_2 + \ldots + \bar{p}_{tm}\bar{\mathbf{x}}_m \text{ for } \bar{x}_t.$$

Then $a(x, \bar{x})$ becomes $b(\mathbf{x}, \bar{\mathbf{x}})$, where $B = PA\bar{P}'$.

Changing now the notation, suppose that when we put

$$\left.\begin{array}{l} p_{t1}\mathbf{x}_1 + p_{t2}\mathbf{x}_2 + \ldots + p_{tm}\mathbf{x}_m \text{ for } x_t \\ q_{t1}\mathbf{y}_1 + q_{t2}\mathbf{y}_2 + \ldots + q_{tm}\mathbf{y}_m \text{ for } y_t \end{array}\right\} (t = 1, 2, \ldots, m),$$

$a(x, y)$ becomes $c(\mathbf{x}, \mathbf{y})$ and $b(x, y)$ becomes $d(\mathbf{x}, \mathbf{y})$.

Then, by Ch. II, § 2, since $PAQ' = C$ and $PBQ' = D$, the determinants

$$\begin{vmatrix} a_{11} - \lambda b_{11} & a_{12} - \lambda b_{12} & \cdot & \cdot & \cdot & a_{1m} - \lambda b_{1m} \\ a_{21} - \lambda b_{21} & a_{22} - \lambda b_{22} & \cdot & \cdot & \cdot & a_{2m} - \lambda b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} - \lambda b_{m1} & a_{m2} - \lambda b_{m2} & \cdot & & \cdot & a_{mm} - \lambda b_{mm} \end{vmatrix}$$

and

$$\begin{vmatrix} c_{11} - \lambda d_{11} & c_{12} - \lambda d_{12} & \cdot & \cdot & \cdot & c_{1m} - \lambda d_{1m} \\ c_{21} - \lambda d_{21} & c_{22} - \lambda d_{22} & \cdot & \cdot & \cdot & c_{2m} - \lambda d_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{m1} - \lambda d_{m1} & c_{m2} - \lambda d_{m2} & \cdot & \cdot & \cdot & c_{mm} - \lambda d_{mm} \end{vmatrix}$$

have the same invariant-factors.

In particular, the second determinant is a multiple of the first.*

* It is the first determinant multiplied by the product of the determinants of P and Q, which are supposed not to vanish.

Ex. If the substitution B is transformed into A by the *orthogonal* substitution P, $a(x, y)$ becomes $b(\mathbf{x}, \mathbf{y})$ when we perform on it the congruent transformation

$$x_t = p_{t1}\mathbf{x}_1 + p_{t2}\mathbf{x}_2 + \ldots + p_{tm}\mathbf{x}_m, \quad y_t = p_{t1}\mathbf{y}_1 + p_{t2}\mathbf{y}_2 + \ldots + p_{tm}\mathbf{y}_m.$$

§ 2. Hermitian Forms.

If A is a Hermitian substitution, so that $a_{ij} = \bar{a}_{ji}$, $a(x, \bar{x})$ is called a *Hermitian form.*

Put

$$\left.\begin{array}{l} p_{t1}\mathbf{x}_1 + p_{t2}\mathbf{x}_2 + \ldots + p_{tm}\mathbf{x}_m \text{ for } x_t \\ \bar{p}_{t1}\bar{\mathbf{x}}_1 + \bar{p}_{t2}\bar{\mathbf{x}}_2 + \ldots + \bar{p}_{tm}\bar{\mathbf{x}}_m \text{ for } \bar{x}_t \end{array}\right\},$$

and $a(x, \bar{x})$ becomes $b(\mathbf{x}, \bar{\mathbf{x}})$, where $B = PA\bar{P}'$.

Since A is Hermitian, by Ch. I, § 10, B is Hermitian, and hence

Any change of variables transforms a Hermitian form into a Hermitian form.

Suppose now that P^{-1} is the unitary substitution (Ch. I, § 12) transforming A into the multiplication

$$x_1' = \lambda_1 x_1, \; x_2' = \lambda_2 x_2, \; \ldots, \; x_m' = \lambda_m x_m.$$

Since P is unitary, $\bar{P}' = P^{-1}$. Therefore $B = PAP^{-1}$, and hence $b(\mathbf{x}, \bar{\mathbf{x}})$ is the Hermitian form

$$\lambda_1 \mathbf{x}_1 \bar{\mathbf{x}}_1 + \lambda_2 \mathbf{x}_2 \bar{\mathbf{x}}_2 + \ldots + \lambda_m \mathbf{x}_m \bar{\mathbf{x}}_m.$$

Hence

Any Hermitian form $a(x, \bar{x})$ with non-zero determinant can be transformed by a suitable unitary change of variables into the form

$$\lambda_1 \mathbf{x}_1 \bar{\mathbf{x}}_1 + \lambda_2 \mathbf{x}_2 \bar{\mathbf{x}}_2 + \ldots + \lambda_m \mathbf{x}_m \bar{\mathbf{x}}_m,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the real characteristic-roots of the associated substitution A.

Suppose now that we have a Hermitian form $a(x, \bar{x})$ whose characteristic-equation

$$\begin{vmatrix} a_{11} - \lambda & . & . & . & a_{1m} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & . & . & . & a_{mm} - \lambda \end{vmatrix} = 0$$

has $m-r$ zero roots, the other roots being $\lambda_1, \lambda_2, \ldots, \lambda_r$.

Suppose $-\epsilon$ is not a root, and let $e(x, \bar{x})$ denote

$$x_1\bar{x}_1 + x_2\bar{x}_2 + \ldots + x_m\bar{x}_m.$$

Then the characteristic-equation of $a(x, \bar{x}) + \epsilon . e(x, \bar{x})$ has the roots

$$\lambda_1 + \epsilon,\ \lambda_2 + \epsilon,\ \ldots,\ \lambda_r + \epsilon,\ \epsilon,\ \epsilon,\ \ldots,\ \epsilon,$$

none of which is zero.

Therefore, as before, a suitable unitary change of variables reduces $a(x, \bar{x}) + \epsilon . e(x, \bar{x})$ to

$$(\lambda_1 \mathbf{x}_1 \bar{\mathbf{x}}_1 + \lambda_2 \mathbf{x}_2 \bar{\mathbf{x}}_2 + \ldots + \lambda_r \mathbf{x}_r \bar{\mathbf{x}}_r) + \epsilon (\mathbf{x}_1 \bar{\mathbf{x}}_1 + \mathbf{x}_2 \bar{\mathbf{x}}_2 + \ldots + \mathbf{x}_m \bar{\mathbf{x}}_m).$$

But a unitary change of variables reduces $e(x, \bar{x})$ to $e(\mathbf{x}, \bar{\mathbf{x}})$, since $e(x, \bar{x})$ is an invariant of a unitary substitution by Ch. I, § 11.

Hence by the unitary change of variables $a(x, \bar{x})$ has been reduced to

$$\lambda_1 \mathbf{x}_1 \bar{\mathbf{x}}_1 + \lambda_2 \mathbf{x}_2 \bar{\mathbf{x}}_2 + \ldots + \lambda_r \mathbf{x}_r \bar{\mathbf{x}}_r.$$

Since $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ are independent linear functions of $x_1, x_2, \ldots, x_m$, the determinants of $a(x, \bar{x})$ and of

$$\lambda_1 x_1 \bar{x}_1 + \lambda_2 x_2 \bar{x}_2 + \ldots + \lambda_r x_r \bar{x}_r$$

have the same rank (§ 1).

But the rank of the latter is r, and therefore the rank of the determinant of A is r. We call r the *rank of the form* $a(x, \bar{x})$. Then we have:—

A Hermitian form of rank r can be reduced by a suitable unitary change of variables into the form

$$\lambda_1 \mathbf{x}_1 \bar{\mathbf{x}}_1 + \lambda_2 \mathbf{x}_2 \bar{\mathbf{x}}_2 + \ldots + \lambda_r \mathbf{x}_r \bar{\mathbf{x}}_r,$$

$\lambda_1, \lambda_2, \ldots, \lambda_r$ *being the real non-zero characteristic-roots of the form.**

Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are positive, and $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_r$ are negative. Putting

$$\sqrt{\lambda_1}\mathbf{x}_1 = \xi_1,\ \ldots,\ \sqrt{\lambda_k}\mathbf{x}_k = \xi_k,\ \sqrt{-\lambda_{k+1}}\mathbf{x}_{k+1} = \xi_{k+1},\ \ldots,\ \sqrt{-\lambda_r}\mathbf{x}_r = \xi_r,$$

we reduce the Hermitian form to

$$\xi_1 \bar{\xi}_1 + \ldots + \xi_k \bar{\xi}_k - \xi_{k+1} \bar{\xi}_{k+1} - \ldots - \xi_r \bar{\xi}_r, \qquad \text{(i)}$$

which is the canonical shape of a Hermitian form of rank r.

If by any other change of variables the Hermitian form $a(x, \bar{x})$ is reduced to

$$\eta_1 \bar{\eta}_1 + \ldots + \eta_l \bar{\eta}_l - \eta_{l+1} \bar{\eta}_{l+1} - \ldots - \eta_s \bar{\eta}_s, \qquad \text{(ii)}$$

* The reduction of the Hermitian form to this canonical shape can be effected more simply if we do not confine ourselves to a unitary change of variables; cf. § 6. By the 'characteristic-roots of the form' we mean the characteristic-roots of the associated substitution.

where $\eta_1, \eta_2, \ldots, \eta_s$ are independent linear functions of $x_1, x_2, \ldots, x_m$; then $r = s$ and $k = l$.

For the rank of the determinant of (ii) is the same as that of the determinant of A; i.e. $s = r$.

We have then

$$\xi_1\bar{\xi}_1 + \ldots + \xi_k\bar{\xi}_k - \xi_{k+1}\bar{\xi}_{k+1} - \ldots - \xi_r\bar{\xi}_r$$
$$\equiv \eta_1\bar{\eta}_1 + \ldots + \eta_l\bar{\eta}_l - \eta_{l+1}\bar{\eta}_{l+1} - \ldots - \eta_r\bar{\eta}_r.$$

Suppose, if possible, $k > l$.

Then we can choose non-zero values of $x_1, x_2, \ldots, x_m$ to make $\eta_1, \eta_2, \ldots, \eta_l, \xi_{k+1}, \ldots, \xi_r$ vanish (for they are $r-k+l$ linear functions of $x_1, x_2, \ldots, x_m$, and $r-k+l < m$), but not all of $\xi_1, \ldots, \xi_k, \eta_{l+1}, \ldots, \eta_r$ vanish.

But then we should have a negative quantity on the right-hand side equal to a positive quantity on the left, which is impossible.

If $r = m$ and $k = m$ or 0, so that $a(x, \bar{x})$ is reducible respectively to

$$\pm(\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \ldots + \xi_m\bar{\xi}_m),$$

$a(x, \bar{x})$ is called a *definite* Hermitian form. It is sometimes called a *positive* Hermitian form when $k = m$, and a *negative* when $k = 0$. It is definite if and only if the characteristic-roots of A are all positive, or all negative. A positive form is always > 0 (and a negative form < 0) for any values of $x_1, x_2, \ldots, x_m$ not all zero.

Since a real symmetric substitution A is a particular case of a Hermitian substitution, we obtain readily in a similar manner :—

A real quadratic form in m variables $a(x, x) \equiv \Sigma a_{ij}x_ix_j$, *where* $a_{ij} = a_{ji}$, *whose determinant is of rank* r, *can be expressed by a suitable orthogonal change of variables in the form*

$$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_r x_r^2,$$

$\lambda_1, \lambda_2, \ldots, \lambda_r$ *being the real non-zero characteristic-roots of the form.**

As before, if a_{ij} is *real* for all values of i and j, $a(x, x)$ can be put in the form

$$\xi_1^2 + \ldots + \xi_k^2 - \xi_{k+1}^2 - \ldots - \xi_r^2,$$

* The reduction of the quadratic form to this canonical shape can be effected more simply if we do not confine ourselves to an orthogonal change of variables; cf. § 6. The reader should notice that among the m^2 terms of $\Sigma a_{ij}x_ix_j$ there are $\frac{1}{2}m(m-1)$ pairs of identical terms such as $a_{ij}x_ix_j$ and $a_{ji}x_ix_j$, where $i \neq j$. We have $\Sigma a_{ij}x_ix_j \equiv a_{11}x_1^2 + a_{22}x_2^2 + \ldots + 2a_{12}x_1x_2 + \ldots$, where the right-hand side contains $\frac{1}{2}m(m+1)$ terms.

where $\xi_1, \ldots, \xi_k, \xi_{k+1}, \ldots, \xi_r$ are independent real linear functions of $x_1, x_2, \ldots, x_m$; and if it can be expressed also in the form $\qquad \eta_1^2 + \ldots + \eta_l^2 - \eta_{l+1}^2 - \ldots - \eta_s^2$,
where $\eta_1, \ldots, \eta_l, \eta_{l+1}, \ldots \eta_s$ are independent real linear functions of $x_1, x_2, \ldots, x_m$, then $r = s$ and $k = l$.

If $r = m$ and $k = m$ or 0, so that $a(x, x)$ is reducible to

$$\pm(\xi_1^2 + \xi_2^2 + \ldots + \xi_m^2),$$

$a(x, x)$ is called a real definite quadratic form, positive or negative according as $k = m$ or $k = 0$.

Ex. 1. In Ch. I, § 12, Ex. 1, the substitution A was transformed by PQ into $(2x, y, -z)$.

The inverse of PQ, which is the same as the substitution transposed to PQ since PQ is orthogonal, has the matrix

$$\begin{vmatrix} -\frac{10}{15} & \frac{11}{15} & \frac{2}{15} \\ \frac{5}{15} & \frac{2}{15} & \frac{14}{15} \\ -\frac{10}{15} & -\frac{10}{15} & \frac{5}{15} \end{vmatrix}.$$

Hence the quadratic form

$$\tfrac{1}{50625}(317x^2 - 142y^2 + 275z^2 - 380yz + 160zx - 212xy)$$

becomes $(2\mathbf{x}^2 + \mathbf{y}^2 - \mathbf{z}^2)$ if we write

$$15x = -10\mathbf{x} + 11\mathbf{y} + 2\mathbf{z}, \quad 15y = 5\mathbf{x} + 2\mathbf{y} + 14\mathbf{z},$$
$$15z = -10\mathbf{x} - 10\mathbf{y} + 5\mathbf{z}.$$

Ex. 2. Transform similarly the quadratic form

$$\tfrac{1}{9}(-x^2 - y^2 + 2z^2 + 4yz + 4zx - 8xy).$$

Ex. 3. If A is a definite Hermitian substitution, so is $\bar{B}'AB$. If A is a real definite symmetric substitution, so is $B'AB$.

[In the first case $\bar{B}'AB$ is Hermitian by Ch. I, § 10. Now by Ch. III, § 1 the forms corresponding to A and $\bar{B}'AB$ can be transformed into one another. But by § 2, if one form is definite, so is the other.

Similarly in the second case.]

Ex. 4. If $a(x, y)$ is a real symmetric bilinear form $(a_{ij} = a_{ji})$, and $a(x, x)$ is a definite quadratic form, then

$$\{a(x, y)\}^2 \leq \{a(x, x)\} \times \{a(y, y)\}.$$

[The equation $a(\lambda x + \mu y, \lambda x + \mu y) = 0$ in λ/μ has unreal roots.]

§ 3. Reduction of a Bilinear Form to Standard Type.

The bilinear form $\Sigma a_{ij} y_i x_j$ may be reduced by an infinity of transformations of variables into the form

$$\mathbf{x}_1\mathbf{y}_1 + \mathbf{x}_2\mathbf{y}_2 + \ldots + \mathbf{x}_r\mathbf{y}_r.$$

If the determinant of the form is not zero, it suffices to put

$\mathbf{x}_t$ for $a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m$ $(t = 1, 2, \ldots, m)$.

If the determinant of the form vanishes, this method breaks down, for then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ would not be independent.

In this case we may proceed as follows:—

First suppose that the quantities $a_{11}, a_{22}, \ldots, a_{mm}$ are not all zero. There is no loss of generality in supposing a_{11} is not zero.

Put $\quad \xi_1$ for $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m$,

and $\quad \eta_1$ for $a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m$.

Then
$$\Sigma\, a_{ij}y_ix_j = \frac{1}{a_{11}}\xi_1\eta_1 + f,$$
where f is a bilinear form in the variables

$x_2, x_3, \ldots, x_m$; $y_2, y_3, \ldots, y_m$ only.

Next suppose that $a_{11} = a_{22} = \ldots = a_{mm} = 0$. There is no loss of generality in supposing $a_{12} \neq 0$.

Now put $x_2 = X_1$, $x_1 = X_2$. Then the coefficient of y_1X_1 is a_{12}, and proceeding as before we get
$$\Sigma\, a_{ij}y_ix_j = \frac{1}{a_{12}}\xi_1\eta_1 + f.$$

The same process can now be applied to f, and so on.

Finally put $\xi_1 = a_{11}\mathbf{x}_1$ (or $\xi_1 = a_{12}\mathbf{x}_1$ in the second case), &c.

By § 1, the determinants of
$$\Sigma\, a_{ij}y_ix_j \text{ and } x_1y_1 + x_2y_2 + \ldots + x_ry_r$$
have the same rank. Hence r is the rank of the determinant of $\Sigma\, a_{ij}y_ix_j$.

Ex. 1. Reduce
$$a(x, y) \equiv y_1(2x_2 - 3x_3) + y_2(4x_1 - 2x_3) + y_3(-3x_1 + x_2)$$
to standard form.

[The coefficient of y_1x_2 is 2. Putting then $x_1 = X_2$, $x_2 = X_1$, we get $a(x, y) \equiv y_1(2X_1 - 3x_3) + y_2(4X_2 - 2x_3) + y_3(X_1 - 3X_2)$.

Put $\xi_1 = 2X_1 - 3x_3$, $\eta_1 = 2y_1 + y_3$, since the coefficient of y_1X_1 is 2, and we have
$$a(x, y) = \tfrac{1}{2}\xi_1\eta_1 + y_2(4X_2 - 2x_3) + y_3(-3X_2 + \tfrac{3}{2}x_3),$$
or
$$a(x, y) = \tfrac{1}{2}\xi_1\eta_1 + \tfrac{1}{4}\xi_2\eta_2,$$
where
$$\xi_2 = 4X_2 - 2x_3, \quad \eta_2 = 4y_2 - 3y_3.$$

Now put $\xi_1 = 2\mathbf{x}_1,\ \xi_2 = 4\mathbf{x}_2,\ \eta_1 = \mathbf{y}_1,\ \eta_2 = \mathbf{y}_2$
and $a(x, y)$ becomes $\mathbf{x}_1\mathbf{y}_1 + \mathbf{x}_2\mathbf{y}_2$, where

$$\mathbf{x}_1 = x_2 - \tfrac{3}{2}x_3,\ \mathbf{x}_2 = x_1 - \tfrac{1}{2}x_3,\ \mathbf{y}_1 = 2y_1 + y_3,\ \mathbf{y}_2 = 4y_2 - 3y_3.]$$

Ex. 2. Reduce to standard form

$$y_1(x_2 - x_3) + y_2(x_1 + 2x_3) + y_3(2x_1 + 4x_2),$$

and $$y_1(3x_1 - 4x_2 - 5x_3 + x_4) + y_2(x_2 + 2x_3 + 2x_4) + y_3(-x_2 - 2x_3 - 2x_4) + y_4(-x_1 - x_3 - 3x_4).$$

Ex. 3. The standard type into which a given bilinear form can be reduced has always the same number of terms, whatever be the manner in which the reduction is performed.

[The number of terms is the rank of the determinant of the form.]

Ex. 4. If a bilinear form is the product of two linear forms, its rank is unity.

Ex. 5. If a bilinear form $\Sigma a_{ij}y_ix_j$ with non-zero determinant is unaltered when we replace

$$\left.\begin{array}{l} x_t \text{ by } p_{t1}x_1 + p_{t2}x_2 + \dots + p_{tm}x_m \\ y_t \text{ by } q_{t1}y_1 + q_{t2}y_2 + \dots + q_{tm}y_m \end{array}\right\} \quad (t = 1, 2, \dots, m),$$

the invariant-factors of P and Q are such that to each invariant-factor $(\lambda - \alpha)^r$ of P corresponds an invariant-factor $(\lambda - \alpha^{-1})^r$ of Q.

[Suppose that $x_1y_1 + x_2y_2 + \dots + x_my_m$ becomes $\Sigma a_{ij}y_ix_j$ when we replace

$$\left.\begin{array}{l} x_t \text{ by } r_{t1}x_1 + r_{t2}x_2 + \dots + r_{tm}x_m \\ y_t \text{ by } s_{t1}y_1 + s_{t2}y_2 + \dots + s_{tm}y_m \end{array}\right\} \quad (t = 1, 2, \dots, m).$$

Then $RES' = A$, $PAQ' = A$. Hence $R^{-1}PR \cdot S'Q'S'^{-1} = E$, or $R^{-1}PR = S'Q'^{-1}S'^{-1}$. Therefore P and Q'^{-1} or P and Q^{-1} have the same invariant-factors.]

§ 4. Transformation of a Bilinear Form into the Sum of Two Bilinear Forms.

In the last section, if $a_{11} \neq 0$ we proved that, when $\Sigma a_{ij}y_ix_j$ is expressed in terms of

$$\xi_1, x_2, x_3, \dots, x_m \text{ and } \eta_1, y_2, y_3, \dots, y_m,$$

where

$$\xi_1 \equiv a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m,\ \eta_1 \equiv a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m,$$

it becomes the sum of a bilinear form in ξ_1, η_1, and a bilinear form in $x_2, x_3, \dots, x_m;\ y_2, y_3, \dots, y_m$.

This is a particular case of the theorem:—

If

$$D \equiv \begin{vmatrix} a_{11} & a_{12} & . & . & . & a_{1k} \\ a_{21} & a_{22} & . & . & . & a_{2k} \\ . & . & . & . & . & . \\ a_{k1} & a_{k2} & . & . & . & a_{kk} \end{vmatrix}$$

is not zero, and $\Sigma a_{ij} y_i x_j$ *is expressed in terms of*

$$\xi_1, \ldots, \xi_k, x_{k+1}, \ldots, x_m; \eta_1, \ldots, \eta_k, y_{k+1}, \ldots, y_m,$$

where

$$\left.\begin{aligned} \xi_t &= a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \\ \eta_t &= a_{1t}y_1 + a_{2t}y_2 + \ldots + a_{mt}y_m \end{aligned}\right\} \quad (t = 1, 2, \ldots, k),$$

then $\Sigma a_{ij} y_i x_j$ *becomes the sum of a bilinear form in*

$$\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k,$$

and a bilinear form in $x_{k+1}, \ldots, x_m, y_{k+1}, \ldots, y_m$.

For $D \times \Sigma a_{ij} y_i x_j$

$$= \begin{vmatrix} a_{11} & a_{12} & . & . & . & a_{1k} & \xi_1 \\ a_{21} & a_{22} & . & . & . & a_{2k} & \xi_2 \\ . & . & . & . & . & . & . \\ a_{k1} & a_{k2} & . & . & . & a_{kk} & \xi_k \\ \eta_1 & \eta_2 & . & . & . & \eta_k & \Sigma a_{ij} y_i x_j \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & . & . & . & a_{1k} & \xi_1 \\ a_{21} & a_{22} & . & . & . & a_{2k} & \xi_2 \\ . & . & . & . & . & . & . \\ a_{k1} & a_{k2} & . & . & . & a_{kk} & \xi_k \\ \eta_1 & \eta_2 & . & . & . & \eta_k & 0 \end{vmatrix}.$$

The second determinant is evidently a bilinear form in $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k$ only; and the first determinant becomes, when we multiply the first k columns by $x_1, x_2, \ldots, x_k$ respectively and subtract from the last column, and multiply the first k rows by $y_1, y_2, \ldots, y_k$ respectively and subtract from the last row,

$$\begin{vmatrix} a_{11} & a_{12} & . & . & . & a_{1k} & u_1 \\ a_{21} & a_{22} & . & . & . & a_{2k} & u_2 \\ . & . & . & . & . & . & . \\ a_{k1} & a_{k2} & . & . & . & a_{kk} & u_k \\ v_1 & v_2 & . & . & . & v_k & f \end{vmatrix},$$

where $\quad u_t \equiv a_{t\,k+1}x_{k+1} + a_{t\,k+2}x_{k+2} + \ldots + a_{tm}x_m,$

$\quad\quad\quad v_t \equiv a_{k+1\,t}y_{k+1} + a_{k+2\,t}y_{k+2} + \ldots + a_{mt}y_m,$

and $\quad f \equiv \Sigma a_{ij} y_i x_j \ (i, j = k+1, k+2, \ldots, m),$

which is a bilinear form in $x_{k+1}, \ldots, x_m, y_{k+1}, \ldots, y_m$.

Dividing by D we get the given theorem.

§ 5. Reduction of a Bilinear Form to Standard Type by a Contragredient Transformation.

The reduction of $\Sigma a_{ij} y_i x_j$ to the form

$$\lambda_1 \mathbf{x}_1 \mathbf{y}_1 + \lambda_2 \mathbf{x}_2 \mathbf{y}_2 + \ldots + \lambda_r \mathbf{x}_r \mathbf{y}_r$$

by the transformation

$$\left.\begin{array}{l} x_t = p_{t1}\mathbf{x}_1 + p_{t2}\mathbf{x}_2 + \ldots + p_{tm}\mathbf{x}_m \\ y_t = q_{t1}\mathbf{y}_1 + q_{t2}\mathbf{y}_2 + \ldots + q_{tm}\mathbf{y}_m \end{array}\right\}, \quad PQ' \equiv E,$$

is only possible if the invariant-factors of A are all linear.

In fact, if the determinant of A does not vanish, we have $PAQ' = PAP^{-1} = M$, where M is the multiplication $x_t' = \lambda_t x_t$.

This is only possible if A has linear invariant-factors. For by Ch. II, § 3, A and M have the same invariant-factors; and the invariant-factors of a multiplication are linear.

If the determinant of A vanishes, we get the same result by applying this argument to the form $(\sum_{i,j} a_{ij} y_i x_j + \epsilon \sum_i x_i y_i)$ and proceeding as in § 2.

Ex. Since by Ch. I, § 9, Ex. 1,

$$(x - 2y - 2z,\ 2x + 3y + 4z,\ -2x - y - 2z)$$

is transformed into the multiplication $(x,\ -y,\ 2z)$ by

$$(x + 2y + 2z,\ -x - y - 2z,\ y + z),$$

whose inverse is $(x - 2z,\ x + y,\ -x - y + z)$, therefore the bilinear form

$$y_1(x_1 - 2x_2 - 2x_3) + y_2(2x_1 + 3x_2 + 4x_3) + y_3(-2x_1 - x_2 - 2x_3)$$

is transformed into $\mathbf{y}_1\mathbf{x}_1 - \mathbf{y}_2\mathbf{x}_2 + 2\mathbf{y}_3\mathbf{x}_3$ by the substitution

$$x_1 = \mathbf{x}_1 - 2\mathbf{x}_3,\quad x_2 = \mathbf{x}_1 + \mathbf{x}_2,\quad x_3 = -\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3,$$
$$y_1 = \mathbf{y}_1 - \mathbf{y}_2,\quad y_2 = 2\mathbf{y}_1 - \mathbf{y}_2 + \mathbf{y}_3,\quad y_3 = 2\mathbf{y}_1 - 2\mathbf{y}_2 + \mathbf{y}_3.$$

§ 6. Reduction of a Bilinear Form to Standard Type by a Congruent Transformation.

If $\Sigma a_{ij} y_i x_j$ is reduced to $(\mathbf{x}_1\mathbf{y}_1 + \mathbf{x}_2\mathbf{y}_2 + \ldots + \mathbf{x}_r\mathbf{y}_r)$ by a *congruent* change of variables

$$\left.\begin{array}{l} x_t = p_{t1}\mathbf{x}_1 + p_{t2}\mathbf{x}_2 + \ldots + p_{tm}\mathbf{x}_m \\ y_t = p_{t1}\mathbf{y}_1 + p_{t2}\mathbf{y}_2 + \ldots + p_{tm}\mathbf{y}_m \end{array}\right\},$$

A must be symmetric.

For if $\mathcal{E}$ * is the substitution corresponding to

$$(x_1 y_1 + x_2 y_2 + \ldots + x_r y_r),$$

* $\mathcal{E}$ has zero determinant unless $r = m$. In this case $\mathcal{E}$ is the unit substitution E.

we have $PAP' = \mathcal{E}$ or $A = P^{-1}\mathcal{E}P'^{-1}$, which is symmetric by Ch. I, § 10.

Conversely,

If $\Sigma a_{ij} y_i x_j$ *is symmetric, we can reduce it to the form*

$$\mathbf{x}_1\mathbf{y}_1 + \mathbf{x}_2\mathbf{y}_2 + \ldots + \mathbf{x}_r\mathbf{y}_r$$

by a congruent change of variables.

First suppose that not all the quantities $a_{11}, a_{22}, a_{33}, \ldots,$ vanish.

There is no loss of generality in supposing $a_{11} \neq 0$.

Then if we put

$$\left.\begin{array}{l}\xi_1 \text{ for } a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m \\ \eta_1 \text{ for } a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m\end{array}\right\},$$

$$\Sigma a_{ij} y_i x_j = \frac{1}{a_{11}} \xi_1 \eta_1 + f,$$

where f only involves the variables

$$x_2, x_3, \ldots, x_m;\ y_2, y_3, \ldots, y_m,$$

and is symmetric.*

If all the quantities $a_{11}, a_{22}, a_{33}, \ldots,$ vanish, there is no loss of generality in supposing $a_{12} \neq 0$.

Put

$X_1 + X_2$ for x_1, $X_1 - X_2$ for x_2, $Y_1 + Y_2$ for y_1, $Y_1 - Y_2$ for y_2.

Then the form $\Sigma a_{ij} y_i x_j$ is tranformed into one in which the coefficient of $X_1 Y_1$ (and of $-X_2 Y_2$) is $a_{12} + a_{21}$ or $2a_{12}$; and we proceed as before.

We can now apply the same process to the symmetric form f, and so on.

Finally we put $\xi_1 / a_{11}^{\frac{1}{2}} = \mathbf{x}_1$, $\eta_1 / a_{11}^{\frac{1}{2}} = \mathbf{y}_1$, &c.

If $\Sigma a_{ij} y_i x_j$ is real as well as symmetric, every transformation in this process is real except perhaps the last. If a_{11} were negative, we should put

$$\xi_1 / (-a_{11})^{\frac{1}{2}} = \mathbf{x}_1,\ \eta_1 / (-a_{11})^{\frac{1}{2}} = \mathbf{y}_1, \text{ \&c.},$$

and thus we reduce $\Sigma a_{ij} y_i x_j$ to the form

$$\mathbf{x}_1\mathbf{y}_1 + \ldots + \mathbf{x}_k\mathbf{y}_k - \mathbf{x}_{k+1}\mathbf{y}_{k+1} - \ldots - \mathbf{x}_r\mathbf{y}_r$$

by a real congruent transformation (cf. § 2).

Taking $y_1 = x_1, y_2 = x_2, \ldots, y_m = x_m$, we reduce the real quadratic form $\Sigma a_{ij} x_i x_j$ ($a_{ij} = a_{ji}$) with determinant of rank r

* The result of operating on a symmetric form with a congruent transformation is symmetric; for if A is symmetric, so is PAP' (Ch. I, § 10). Similarly the result of operating on an alternate form with a congruent transformation is alternate.

to the sum of squares $(\mathbf{x}_1^2+\mathbf{x}_2^2+\ldots+\mathbf{x}_r^2)$; or to the form $(\mathbf{x}_1^2+\ldots+\mathbf{x}_k^2-\mathbf{x}_{k+1}^2-\ldots-\mathbf{x}_r^2)$, if we confine ourselves to real transformations.

By a similar process the Hermitian form $\Sigma a_{ij}\bar{x}_i x_j\ (a_{ij}=\bar{a}_{ji})$ can be transformed into

$$\mathbf{x}_1\bar{\mathbf{x}}_1+\ldots+\mathbf{x}_k\bar{\mathbf{x}}_k-\mathbf{x}_{k+1}\bar{\mathbf{x}}_{k+1}-\ldots-\mathbf{x}_r\bar{\mathbf{x}}_r,^*$$

taking $y_1=\bar{x}_1,\ y_2=\bar{x}_2,\ \ldots,\ y_m=\bar{x}_m$. For an alternative method, see § 2.

The proof of § 4 shows that the quadratic form

$$\Sigma\, a_{ij}x_i x_j\ (a_{ij}=a_{ji})$$

is reduced to the sum of a quadratic form in $\xi_1, \xi_2, \ldots, \xi_k$ and a quadratic form in $x_{k+1}, x_{k+2}, \ldots, x_m$ by the substitution

$$\xi_t = a_{t1}x_1+a_{t2}x_2+\ldots+a_{tm}x_m\ (t=1, 2, \ldots, k),$$

provided
$$\begin{vmatrix} a_{11} & a_{12} & . & . & . & a_{1k} \\ a_{21} & a_{22} & . & . & . & a_{2k} \\ . & . & . & . & . & . \\ a_{k1} & a_{k2} & . & . & . & a_{kk} \end{vmatrix} \neq 0.$$

Similarly the Hermitian form $\Sigma\, a_{ij}\bar{x}_i x_j\ (a_{ij}=\bar{a}_{ji})$ is reduced to the sum of a Hermitian form in $\xi_1, \xi_2, \ldots, \xi_k$ and a Hermitian form in $x_{k+1}, x_{k+2}, \ldots, x_m$ under the same condition.

Ex. 1. $y^2+8z^2+37w^2-6yz+4zx-2xy-10xw+12yw-34zw$
$$= (-x+y-3z+6w)^2-x^2-z^2+w^2+2xw+2zw$$
$$= (-x+y-3z+6w)^2-(x+z-w)^2+2w^2$$
$$= (-x+y-3z+6w)^2+(\sqrt{2}w)^2-(x+z-w)^2.$$

The given quadratic form is also equivalent to

$$(x+z-3w)^2+(\sqrt{2}\,\{y-2z+4w\})^2-(x+y-z+2w)^2,$$

which is again the sum of two real squares less one real square.

Ex. 2. $yz+zx+2xy$.

Here the coefficients of x^2, y^2, z^2 are all zero.

Put $x=X+Y,\ y=X-Y$. Then

$$yz+zx+2xy = 2X^2-2Y^2+2Xz = 2(X+\tfrac{1}{2}z)^2-2Y^2-\tfrac{1}{2}z^2$$
$$= \left(\frac{\sqrt{2}}{2}\{x+y+z\}\right)^2-\left(\frac{\sqrt{2}}{2}\{x-y\}\right)^2-\left(\frac{\sqrt{2}}{2}z\right)^2.$$

* When $a_{11}=a_{22}=\ldots=a_{mm}=0$, $a_{12}\neq 0$; put $x_1=X_1+iX_2$, $x_2=iX_1+X_2$ (instead of $x_1=X_1+X_2$, $x_2=X_1-X_2$) if $a_{12}+a_{21}=0$; and $x_1=X_1+a_{12}X_2$, $x_2=-X_1+a_{21}X_2$ if $a_{12}+a_{21}\neq 0$.

Ex. 3. Express as the sum of squares

$$yz+zx+xy-xw-yw-zw,$$
$$xz+xu-yu+zw+wu-yz,$$
$$x^2+4y^2+z^2+4xy-2xz.$$

Ex. 4. $\bar{x}[iy+(1+i)z]+\bar{y}[-ix+(1-2i)z]$
$$+\bar{z}[(1-i)x+(1+2i)y].$$

Put $x=X+iY$, $y=iX+Y$. The given Hermitian form becomes $\bar{X}[-2X-z]+\bar{Y}[2Y+(2-3i)z]+\bar{z}[-X+(2+3i)Y]$.

Put $\xi=-2X-z$ and it becomes
$$-\tfrac{1}{2}\xi\bar{\xi}+\bar{Y}[2Y+(2-3i)z]+\bar{z}[(2+3i)Y+\tfrac{1}{2}z].$$

Put $\eta=2Y+(2-3i)z$ and it becomes $-\frac{1}{2}\xi\bar{\xi}+\frac{1}{2}\eta\bar{\eta}-6z\bar{z}$.

Put $\xi=\sqrt{2}\mathbf{y}$, $\eta=\sqrt{2}\mathbf{x}$, $z=\dfrac{1}{\sqrt{6}}\mathbf{z}$ and it becomes $\mathbf{x}\bar{\mathbf{x}}-\mathbf{y}\bar{\mathbf{y}}-\mathbf{z}\bar{\mathbf{z}}$.

Ex. 5. Transform into the standard type
$$\bar{x}[5x-iy+(1-2i)z]+\bar{y}[ix+y+iz]+\bar{z}[(1+2i)x-iy+z],$$
and
$$i\{(y\bar{z}-\bar{y}z)+(z\bar{x}-\bar{z}x)+(x\bar{y}-\bar{x}y)\}.$$

Ex. 6. A quadratic form is the product of two linear factors if and only if its determinant is of rank 1 or 2.

Ex. 7. Evaluate $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Ve^{-U}\,dx\,dy\,dz$, where U and V are real quadratic forms in x, y, z, a, and $U+ka^2$ is always positive or zero; and similarly for any number of variables.

For instance, evaluate
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2-2xy+3y^2+4x+4y+16)e^{-U}\,dx\,dy,$$
when
$$U\equiv x^2-2xy+2y^2+4x+9.$$

[Express the integral in terms of $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, where
$$U\equiv\mathbf{x}^2+\mathbf{y}^2+\mathbf{z}^2-ka^2.$$
Then use
$$\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi},\quad \int_{-\infty}^{\infty}xe^{-x^2}dx=0,\quad \int_{-\infty}^{\infty}x^2e^{-x^2}dx=\tfrac{1}{2}\sqrt{\pi}.$$
In the illustration take
$$a=1,\ k=-1,\ \mathbf{x}=x-y+2,\ \mathbf{y}=y+2.]$$

§ 7. Reduction of an Alternate Form.

The alternate form $\Sigma a_{ij}y_ix_j$ $(a_{ij}=-a_{ji})$ *can be reduced to the form*
$$(\mathbf{x}_1\mathbf{y}_2-\mathbf{x}_2\mathbf{y}_1)+(\mathbf{x}_3\mathbf{y}_4-\mathbf{x}_4\mathbf{y}_3)+(\mathbf{x}_5\mathbf{y}_6-\mathbf{x}_6\mathbf{y}_5)+\dots$$
by a congruent change of variables.

Since the form is alternate, $a_{11} = a_{22} = a_{33} = \ldots = 0$.

We may suppose without loss of generality that $a_{12} \neq 0$.

Put
$$\left.\begin{array}{ll} \xi_1 \text{ for } & a_{12}x_2 + a_{13}x_3 + \ldots + a_{1m}x_m \\ \eta_1 \text{ for } & -a_{21}y_2 - a_{31}y_3 - \ldots - a_{m1}y_m \\ \xi_2 \text{ for } & a_{21}x_1 + a_{23}x_3 + \ldots + a_{2m}x_m \\ \eta_2 \text{ for } & -a_{12}y_1 - a_{32}y_3 - \ldots - a_{m2}x_m \end{array}\right\}.$$

This transformation is congruent, since $a_{ij} = -a_{ji}$.

Then
$$\Sigma a_{ij}y_ix_j = \frac{1}{a_{21}}(\xi_1\eta_2 - \xi_2\eta_1) + f,$$
where f is a bilinear form in
$$x_3, x_4, \ldots, x_m;\ y_3, y_4, \ldots, y_m$$
only, and is alternate.*

The same process may now be applied to f, and so on.

Finally put
$$\xi_1 = a_{21}{}^{\frac{1}{2}}\mathbf{x}_1,\ \xi_2 = a_{21}{}^{\frac{1}{2}}\mathbf{x}_2,\ \eta_1 = a_{21}{}^{\frac{1}{2}}\mathbf{y}_1,\ \eta_2 = a_{21}{}^{\frac{1}{2}}\mathbf{y}_2, \text{ \&c.}$$

This last transformation is unreal if a_{21} is not real and positive. If a_{ij} is always real, we can perform the reduction by a real transformation, by taking
$$\xi_1 = a_{12}{}^{\frac{1}{2}}\mathbf{x}_2,\ \xi_2 = a_{12}{}^{\frac{1}{2}}\mathbf{x}_1,\ \eta_1 = a_{12}{}^{\frac{1}{2}}\mathbf{y}_2,\ \eta_2 = a_{12}{}^{\frac{1}{2}}\mathbf{y}_1$$
when a_{21} is negative, and so on.

Ex. 1. $y_1(x_2 - 2x_3 + 3x_4) + y_2(-x_1 - x_3 + x_4) + y_3(2x_1 + x_2 + 2x_4) + y_4(-3x_1 - x_2 - 2x_3)$.

Put
$$\xi_1 = x_2 - 2x_3 + 3x_4,\ \eta_1 = y_2 - 2y_3 + 3y_4,$$
$$\xi_2 = -x_1 - x_3 + x_4,\ \eta_2 = -y_1 - y_3 + y_4,$$
and the given alternate bilinear form is
$$-(\xi_1\eta_2 - \xi_2\eta_1) + (y_3x_4 - y_4x_3) = (\mathbf{x}_1\mathbf{y}_2 - \mathbf{x}_2\mathbf{y}_1) + (\mathbf{x}_3\mathbf{y}_4 - \mathbf{x}_4\mathbf{y}_3),$$
where
$$\mathbf{x}_1 = \xi_2 = (-x_1 - x_3 + x_4),\ \mathbf{x}_2 = \xi_1 = (x_2 - 2x_3 + 3x_4),\ \mathbf{x}_3 = x_4,\ \mathbf{x}_4 = x_3,$$
and
$$\mathbf{y}_1 = \eta_2 = (-y_1 - y_3 + y_4),\ \mathbf{y}_2 = \eta_1 = (y_2 - 2y_3 + 3y_4),\ \mathbf{y}_3 = y_4,\ \mathbf{y}_4 = y_3.$$

Ex. 2. Reduce to standard form
$$y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2)$$
and
$$y_1(x_2 - x_3 + 2x_4) + y_2(-x_1 + x_3 + x_4) + y_3(x_1 - x_2 + 3x_4) + y_4(-2x_1 - x_2 - 3x_3).$$

* See footnote, p. 73.

§ 8. Adjoint Bilinear Forms.

The bilinear form

$$-\begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} & x_1 \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} & x_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mm} & x_m \\ y_1 & y_2 & \cdot & \cdot & \cdot & y_m & 0 \end{vmatrix} \equiv \Sigma A_{ij} x_i y_j$$

is called the form *adjoint* to $\Sigma a_{ij} y_i x_j$.

A_{ij} is the co-factor of a_{ij} in the determinant $|a|$ of $\Sigma a_{ij} y_i x_j$.

If the rank of $\Sigma a_{ij} y_i x_j$ is $\leq m-2$, each of the quantities A_{ij} vanishes, and therefore the adjoint form vanishes identically.

If the rank of $\Sigma a_{ij} y_i x_j$ is m, the rank of the adjoint form is also m; for its determinant is $\{|a|\}^{m-1}$.*

The adjoint form of the adjoint form is the original bilinear form multiplied by $\{|a|\}^{m-2}$.*

If the rank of $\Sigma a_{ij} y_i x_j$ is $m-1$, the rank of the adjoint form is 1.

For not all the quantities A_{ij} vanish while

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = \begin{vmatrix} a_{33} & \cdot & \cdot & \cdot & a_{3m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m3} & \cdot & \cdot & \cdot & a_{mm} \end{vmatrix} \times |a| = 0,\text{* \&c.}$$

Since the adjoint form is of rank 1, it is the product of two linear factors (§ 3).

Suppose it is

$$(p_1 x_1 + p_2 x_2 + \ldots + p_m x_m)(q_1 y_1 + q_2 y_2 + \ldots + q_m y_m),$$

where $p_s \neq 0$, $q_t \neq 0$.

Comparing coefficients of $y_t x_1, y_t x_2, \ldots, y_t x_m$, we get

$$p_1 : p_2 : \ldots : p_m = A_{1t} : A_{2t} : \ldots : A_{mt},$$

and similarly

$$q_1 : q_2 : \ldots : q_m = A_{s1} : A_{s2} : \ldots : A_{sm}.$$

Since $|a| = 0$, $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_m$ are the solutions of the equations

* See, for instance, Burnside and Panton's *Theory of Equations*, Ch. XI.

$$\frac{\partial a}{\partial x_1} = \frac{\partial a}{\partial x_2} = \ldots = \frac{\partial a}{\partial x_m} = 0, \text{ and } \frac{\partial a}{\partial y_1} = \frac{\partial a}{\partial y_2} = \ldots = \frac{\partial a}{\partial y_m} = 0$$

respectively; where $a \equiv \Sigma a_{ij} y_i x_j$.

Putting $y_1 = x_1$, $y_2 = x_2$, ..., $y_m = x_m$, $a_{ij} = a_{ji}$ in the bilinear form and its adjoint form, we obtain a quadratic form and the *adjoint quadratic form.*

If the rank of the quadratic form a is $\leq m-2$, the adjoint form vanishes identically.

If the rank of a is m, the rank of the adjoint form is also m; and the adjoint form of the adjoint form is the original quadratic form multiplied by $\{|a|\}^{m-2}$.

If the rank of a is $m-1$, the adjoint form is a perfect square $(p_1 x_1 + p_2 x_2 + \ldots + p_m x_m)^2$, where $p_1, p_2, \ldots, p_m$ are the solutions of

$$\frac{\partial a}{\partial x_1} = \frac{\partial a}{\partial x_2} = \ldots = \frac{\partial a}{\partial x_m}.$$

Similarly for a Hermitian form.

Ex. Find the adjoint forms of

$y_1(x_1+x_2)+y_2(x_1+2x_2)$ and of $4x^2-2y^2-4z^2+6yz+6zx-2xy$.

[$y_1(2x_1-x_2)+y_2(-x_1+x_2)$ and $-(x-5y-3z)^2$.]

§ 9. The Sum of Substitutions.

Let $a(x, y)$, $b(x, y)$, ..., $k(x, y)$ be the bilinear forms corresponding to the substitutions $A, B, \ldots, K$; and let S be the substitution corresponding to the bilinear form

$$s(x, y) \equiv \alpha \,.\, a(x, y) + \beta \,.\, b(x, y) + \ldots + \kappa \,.\, k(x, y),$$

so that

$$s_{ij} = \alpha a_{ij} + \beta b_{ij} + \ldots + \kappa k_{ij} \ (i, j = 1, 2, \ldots, m).$$

Then we say that

$$S = \alpha A + \beta B + \ldots + \kappa K.$$

If $s(x, y)$ vanishes identically, we say that

$$\alpha A + \beta B + \ldots + \kappa K = 0.$$

In this case $A, B, \ldots, K$ are said to be *linearly dependent.**

It follows at once from the definition that

$$P(\alpha A + \beta B) = \alpha PA + \beta PB, \ P(\alpha A + \beta B + \gamma C) = \alpha PA + \beta PB + \gamma PC, \text{ and so on.}$$

* If we cannot choose $\alpha, \beta, \ldots, \kappa$ to make $\alpha A + \beta B + \ldots + \kappa K = 0$, $A, B, \ldots, K$ are 'linearly independent'.

Similarly

$$(\alpha A+\beta B+\ldots+\kappa K)Q=\alpha AQ+\beta BQ+\ldots+\kappa KQ.$$

Again,

$$P(\alpha A+\beta B+\ldots+\kappa K)Q=P(\alpha AQ+\beta BQ+\ldots+\kappa KQ)$$
$$=\alpha PAQ+\beta PBQ+\ldots+\kappa PKQ.$$

Ex. 1. If A, B, C, ..., K are m^2 linearly independent substitutions of degree m, any other substitution S of degree m is linearly dependent on A, B, C, ..., K.

[We can always find m^2 quantities α, β, γ, ..., κ to satisfy

$$s_{ij}=\alpha a_{ij}+\beta b_{ij}+\ldots+\kappa k_{ij}\ (i,\ j=1,\ 2,\ \ldots,\ m).]$$

Ex. 2. The powers $A^0(\equiv E)$, $A^1(\equiv A)$, A^2, A^3, ... of a given substitution A cannot be all linearly independent.

[Use the last Example.]

Ex. 3. If A^r is linearly dependent on A^0, A^1, A^2, A^3, ..., A^{r-1}, so are A^{r+1}, A^{r+2}, A^{r+3}, ...

[If $A^r=a_0A^0+a_1A^1+\ldots+a_{r-1}A^{r-1}$, $A^{r+1}=a_0A^1+a_1A^2+\ldots$
$+a_{r-1}A^r=a_{r-1}A^0+(a_0+a_{r-1}a_1)A^1+(a_1+a_{r-1}a_2)A^2+\ldots$
$+(a_{r-2}+a^2_{r-1})A^{r-1}$.]

Ex. 4. If A^0, A^1, A^2, ..., A^r are linearly dependent, but A^0, A^1, A^2, ..., A^l are linearly independent ($l<r$), there cannot be two distinct linear relations between A^0, A^1, A^2, ..., A^r.

[Elimination of A^r between them would give an equation of lower degree.]

Ex. 5. $A+A'$ is symmetric; $A-A'$ is alternate.

Ex. 6. Any substitution can be expressed in one and only one way as the sum of a symmetric and an alternate substitution.

[$A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')$.]

Ex. 7. If A is real, $AA'\neq 0$ unless $A=0$.

[Consider the leading diagonal of AA'.]

Ex. 8. If each of the substitutions A, B, C, ... is permutable with each of P, Q, R, ..., any linear function of A, B, C, ... is permutable with any linear function of P, Q, R, ...

§ 10. Relations between the Powers of a Substitution.

(I) Suppose that we have a relation

$$p_0A^r+p_1A^{r-1}+p_2A^{r-2}+\ldots+p_{r-1}A^1+p_rA^0=0$$

between the powers of a substitution A.*

Then we have a similar relation between the powers of any substitution into which A can be transformed.

* $A^0\equiv E$, $A^1\equiv A$.

For if $S^{-1}AS=B$,

$$\begin{aligned}0 &= S^{-1}(p_0A^r+p_1A^{r-1}+\dots+p_{r-1}A^1+p_rA^0)S\\ &= p_0S^{-1}A^rS+p_1S^{-1}A^{r-1}S+\dots+p_{r-1}S^{-1}A^1S+p_rS^{-1}A^0S\\ &= p_0B^r+p_1B^{r-1}+\dots+p_{r-1}B^1+p_rB^0,\end{aligned}$$

by § 9.

(II) Again, we see at once from § 9 that if

$$(p_0x^r+p_1x^{r-1}+\dots+p_{r-1}x+p_r)(q_0x^s+q_1x^{s-1}+\dots+q_{s-1}x+q_s)$$
$$\equiv e_0x^{r+s}+e_1x^{r+s-1}+\dots+e_{r+s},$$

then $\quad e_0A^{r+s}+e_1A^{r+s-1}+\dots+e_{r+s-1}A^1+e_{r+s}A^0=0.$

(III) Yet again, we readily prove that if both

$$p_0A^r+p_1A^{r-1}+\dots+p_{r-1}A^1+p_rA^0=0, \qquad \text{(i)}$$

and
$$q_0A^s+q_1A^{s-1}+\dots+q_{s-1}A^1+q_sA^0=0, \qquad \text{(ii)}$$

then
$$b_0A^l+b_1A^{l-1}+\dots+b_{l-1}A^1+b_lA^0=0,$$

where $b_0x^l+b_1x^{l-1}+\dots+b_l$ is the highest common factor of

$$p_0x^r+p_1x^{r-1}+\dots+p_r \text{ and } q_0x^s+q_1x^{s-1}+\dots+q_s.$$

(IV) It follows at once that if both (i) and (ii) are true, but $A^0, A^1, A^2, \dots, A^t$ are linearly independent when $t<r$, so that (i) is the equation of lowest degree satisfied by A, then $q_0x^s+q_1x^{s-1}+\dots+q_s$ is a multiple of $p_0x^r+p_1x^{r-1}+\dots+p_r$.

(V) We readily deduce that, if A is a direct product whose constituents $B, C, \dots$ satisfy equations

$$b_0B^\beta+b_1B^{\beta-1}+\dots+b_\beta B^0=0,\; c_0C^\gamma+c_1C^{\gamma-1}+\dots+c_\gamma C^0=0, \dots$$

of lowest degrees; then the equation of lowest degree satisfied by A is

$$p_0A^r+p_1A^{r-1}+\dots+p_rA^0=0,$$

where
$$p_0x^r+p_1x^{r-1}+\dots+p_r$$

is the least common multiple of

$$b_0x^\beta+b_1x^{\beta-1}+\dots+b_\beta,\; c_0x^\gamma+c_1x^{\gamma-1}+\dots+c_\gamma, \dots.$$

(VI) We now show that if A is the substitution of Ch. II, § 5, Corollary I, with invariant-factors

$$\begin{array}{ll}(\lambda-\alpha)^{a_1}, (\lambda-\alpha)^{a_2}, (\lambda-\alpha)^{a_3} \dots, & \text{where } a_1\geq a_2\geq a_3\geq\dots,\\ (\lambda-\beta)^{b_1}, (\lambda-\beta)^{b_2}, (\lambda-\beta)^{b_3} \dots, & \text{where } b_1\geq b_2\geq b_3\geq\dots,\\ (\lambda-\gamma)^{c_1}, (\lambda-\gamma)^{c_2}, (\lambda-\gamma)^{c_3} \dots, & \text{where } c_1\geq c_2\geq c_3\geq\dots,\\ \dots\dots & \dots\dots\end{array}$$

then the equation of lowest degree satisfied by A is

$$p_0A^r + p_1A^{r-1} + p_2A^{r-2} + \ldots + p_{r-1}A^1 + p_r = 0,$$

where

$$p_0(\lambda-\alpha)^{a_1}(\lambda-\beta)^{b_1}(\lambda-\gamma)^{c_1}\ldots \equiv p_0\lambda^r + p_1\lambda^{r-1} + p_2\lambda^{r-2} + \ldots + p_r,$$

so that

$$r = a_1 + b_1 + c_1 + \ldots.$$

By the preceding argument (I) it is sufficient to prove that the theorem is true for the canonical substitution N of Ch. II, § 5, into which A can be transformed.

Now by (II), if we can prove that

$$(P-\alpha)^m \equiv P^m - {}^mC_1\alpha P^{m-1} + {}^mC_2\alpha^2P^{m-2} - \ldots + (-1)^m\alpha^mP^0 = 0,$$

where P is

$$x_1' = \alpha x_1 + x_2,\ x_2' = \alpha x_2 + x_3,\ \ldots,\ x'_{m-1} = \alpha x_{m-1} + x_m,\ x_m' = \alpha x_m;$$

then $(Q-\alpha)^m = 0$, if Q is

$$x_1' = \alpha x_1 + x_2,\ x_2' = \alpha x_2 + x_3, \ldots,\ x_n' = \alpha x_{n-1} + x_n,\ x_n' = \alpha x_n,$$

where $n \leq m$.

Hence by (V) it is sufficient to prove that $(P-\alpha)^m = 0$ is the equation of least degree satisfied by P in order to prove the property of A referred to.

Suppose P satisfies the equation

$$e_0P^t + e_1P^{t-1} + \ldots + e_{t-1}P^1 + e_tP^0 = 0.$$

Then, using the value of P^t given in Ch. I, § 8, we have the m distinct equations

$$\left.\begin{array}{rl} 0 = & e_t + \alpha e_{t-1} + \alpha^2 e_{t-2} + \ldots + \alpha^t e_0 \\ 0 = & e_{t-1} + {}^2C_1\alpha e_{t-2} + \ldots + {}^tC_1\alpha^{t-1}e_0 \\ 0 = & e_{t-2} + \ldots + {}^tC_2\alpha^{t-2}e_0 \\ & \ldots\ldots\ldots\ldots \end{array}\right\}.$$

We see immediately that these cannot be satisfied by non-zero values of $e_0, e_1, \ldots, e_t$ when $t < m$; but that they are satisfied by

$$t = m,\ e_0 = 1,\ e_1 = -{}^mC_1\alpha,\ e_2 = {}^mC_2\alpha^2,\ \ldots,\ e_m = (-1)^m\alpha^m,$$

since

$${}^mC_p - {}^mC_1{}^{m-1}C_p + {}^mC_2{}^{m-2}C_p - {}^mC_3{}^{m-3}C_p + \ldots$$
$$= \frac{m!}{p!}(1 - {}^pC_1 + {}^pC_2 - {}^pC_3 + \ldots) = 0.$$

Ex. 1. If the characteristic-equation of A is

$$\lambda^m + a_1\lambda^{m-1} + a_2\lambda^{m-2} + \dots + a_m = 0,$$

then $$A^m + a_1 A^{m-1} + a_2 A^{m-2} + \dots + a_m A^0 = 0.$$

[This is the 'Hamilton-Cayley' relation. It follows at once from § 10 (II).]

Ex. 2. The coefficients of the equation of least degree satisfied by A are rational functions of the coefficients of A.

[The equation is found by equating to zero the quotient of the characteristic-determinant of A by the least common multiple of its first minors.]

Ex. 3. If $p_0x^r + p_1x^{r-1} + \dots + p_{r-1}x + p_r = 0$ has r unequal roots, and $p_0A^r + p_1A^{r-1} + \dots + p_{r-1}A^1 + p_rA^0 = 0$, the invariant-factors of A are all linear.

Ex. 4. Prove the result of § 10 (VI) by using the canonical form of Ch. II, § 6.

Ex. 5. If λ is a characteristic-root of A,

$$p_0\lambda^r + p_1\lambda^{r-1} + \dots + p_{r-1}\lambda + p_r$$

is a characteristic-root of

$$p_0A^r + p_1A^{r-1} + \dots + p_{r-1}A^1 + p_rA^0.$$

[It is sufficient to prove this for a canonical substitution.]

CHAPTER IV

APPLICATIONS

§ 1.

IN this chapter we give a few applications of the results obtained heretofore.* For further details we must refer the reader to the works mentioned in the preface.

The Solution of Simultaneous Linear Differential Equations with Constant Coefficients.

Suppose we have the equations

$$\left.\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m + f_1(t) \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m + f_2(t) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dot{x}_m &= a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mm}x_m + f_m(t) \end{aligned}\right\}, \quad \dots\dots\dots \text{(i)}$$

where the dots denote differentiation with respect to a variable t of which $x_1, x_2, \dots, x_m$ are functions, while the a's are constants independent of t.

We can choose homogeneous linear functions

$$\xi_1, \xi_2, \dots, \xi_m \text{ of } x_1, x_2, \dots, x_m$$

such that, when we express $x_1, x_2, \dots, x_m$ in terms of $\xi_1, \xi_2, \dots, \xi_m$ the equations (i) take the 'canonical' form

$$\dot{\xi}_i = \lambda_i \xi_i + \beta_i \xi_{i+1} + F_i(t) \quad (i = 1, 2, \dots, m), \dots\dots\dots \text{(ii)}$$

where $\beta_i = 0$ or 1 and is certainly zero if $\lambda_i \neq \lambda_{i+1}$ (Ch. I, § 9.)

The equations (ii) are at once solved. For instance, suppose the first four of them are

$$\dot{\xi}_1 = \alpha\xi_1 + \xi_2 + F_1(t), \quad \dot{\xi}_2 = \alpha\xi_2 + \xi_3 + F_2(t),$$
$$\dot{\xi}_3 = \alpha\xi_3 + \xi_4 + F_3(t), \quad \dot{\xi}_4 = \alpha\xi_4 + F_4(t).$$

Then we have

$$\xi_1 = e^{\alpha t}\left\{\int e^{-\alpha t}F_1(t)\,dt + \iint e^{-\alpha t}F_2(t)\,dt^2 + \iiint e^{-\alpha t}F_3(t)\,dt^3 \right.$$
$$\left. + \iiiint e^{-\alpha t}F_4(t)\,dt^4 + K_0 + K_1\frac{t}{1\,!} + K_2\frac{t^2}{2\,!} + K_3\frac{t^3}{3\,!}\right\},$$

* See also Ch. I, § 10, Ex. 11, § 13, Ex. 4, 5; Ch. III, § 6, Ex. 7.

$$\xi_2 = e^{\alpha t}\left\{\int e^{-\alpha t}F_2(t)\,dt + \iint e^{-\alpha t}F_3(t)\,dt^2 + \iiint e^{-\alpha t}F_4(t)\,dt^3 + K_1 + K_2\frac{t}{1!} + K_3\frac{t^2}{2!}\right\},$$

$$\xi_3 = e^{\alpha t}\left\{\int e^{-\alpha t}F_3(t)\,dt + \iint e^{-\alpha t}F_4(t)\,dt^2 + K_2 + K_3\frac{t}{1!}\right\},$$

$$\xi_4 = e^{\alpha t}\{\int e^{-\alpha t}F_4(t)\,dt + K_3\},$$

where K_1, K_2, K_3, K_4 are the arbitrary constants of integration. When $\xi_1, \xi_2, \ldots, \xi_m$ are thus expressed in terms of t, $x_1, x_2, \ldots, x_m$ can at once be expressed in terms of t.

In practice, the following method of solving equations (i) is usually easier.

Let $(X_1, X_2, \ldots, X_m)$ be a pole of the substitution A'

$$x_i' = a_{1i}x_1 + a_{2i}x_2 + \ldots + a_{mi}x_m, \quad (i = 1, 2, \ldots, m)$$

corresponding to a characteristic-root λ. Multiply equations (i) by $X_1, X_2, \ldots, X_m$ respectively and add. We get

$$\dot{\mathbf{x}} = \lambda\mathbf{x} + f(t), \qquad \text{(iii)}$$

where $\mathbf{x} = X_1x_1 + X_2x_2 + \ldots + X_mx_m$ *

and $f(t) = X_1 . F_1(t) + X_2 . F_2(t) + \ldots + X_m . F_m(t)$.

This gives $\mathbf{x} = e^{\lambda t}\{\int e^{-\lambda t}f(t)\,dt + K\}$ .... (iv)

If A' (and therefore A) has r distinct poles, we get r equations such as (iii), enabling us to reduce the system (i) to a system of equations in $x_1, x_2, \ldots, x_{m-r}$. We may then apply the same process to this new system. If $r = m$, the equations (iv) give $x_1, x_2, \ldots, x_m$ in terms of t directly.

The system of equations

$$b_{i1}\dot{x}_1 + b_{i2}\dot{x}_2 + \ldots + b_{im}\dot{x}_m = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{im}x_m + f_i(t) \quad (i = 1, 2, \ldots, m)$$

can be reduced to a system such as (i) by solving algebraically for $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_m$; or we may obtain an equation such as (iii) if we choose $\lambda, X_1, X_2, \ldots, X_m$ to satisfy

$$(a_{1i} - \lambda b_{1i})X_1 + (a_{2i} - \lambda b_{2i})X_2 + \ldots + (a_{mi} - \lambda b_{mi})X_m = 0 \quad (i = 1, 2, \ldots, m).$$

Elimination of $X_1, X_2, \ldots, X_m$ gives an equation of the m-th degree for λ; and when λ is obtained, we immediately

* The quantity $\mathbf{x}$ is sometimes called a 'principal coordinate'; see § 10.

deduce $X_1 : X_2 : \ldots : X_m$. Another method of solving equations (i) is suggested in Ex. 9 to 12 below.

Any simultaneous linear differential equations with constant coefficients can be put into the form (i).

Suppose, for instance, we have m such linear equations connecting $x_1, x_2, \ldots, x_m$; $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_m$; $\ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_m$; $\dddot{x}_1, \dddot{x}_2, \ldots, \dddot{x}_m$.

In these m equations put p_1 for $\dot{x}_1, \ldots, p_m$ for $\dot{x}_m$; r_1 for $\ddot{x}_1, \ldots, r_m$ for $\ddot{x}_m$; $\dot{r}_1$ for $\dddot{x}_1, \ldots, \dot{r}_m$ for $\dddot{x}_m$.

Solving the equations, we express $\dot{r}_1, \dot{r}_2, \ldots, \dot{r}_m$ linearly in terms of $x_1, \ldots, x_m$; $p_1, \ldots, p_m$; $r_1, \ldots, r_m$.

The m equations thus found form, together with

$$\dot{x}_1 = p_1, \dot{x}_2 = p_2, \ldots, \dot{x}_m = p_m; \dot{p}_1 = r_1, \dot{p}_2 = r_2, \ldots, \dot{p}_m = r_m,$$

a system of the type (i), which can be solved as before to give $x_1, x_2, \ldots, x_m$ in terms of t.

Ex. 1. Solve the equations

$$\dot{x} = x - y + 2z - 2w,\ \dot{y} = -3y - 2w,\ \dot{z} = -2x + 2y - 3z + 3w,$$
$$\dot{w} = 2y + w.$$

[By Ch. I, § 9, Ex. 3, if

$$\xi = 3x + 2z,\ \eta = 2x + y + 2z,\ \zeta = x + z,\ \omega = y + w,$$

we have $\dot{\xi} = -\xi + \eta,\ \dot{\eta} = -\eta,\ \dot{\zeta} = -\zeta + \omega,\ \dot{\omega} = -\omega$;

$\therefore\ 3x + 2z = (A + Bt)e^{-t},\ 2x + y + 2z = Be^{-t},$

$x + z = (C + Dt)e^{-t},\ y + w = De^{-t}$.]

Ex. 2. Solve the equations

$$\dot{x} = 3x + 4y + z,\ \dot{y} = -y + z,\ \dot{z} = -x - 3y + z.$$

[Multiply by a, b, c, and add; where (a, b, c) is a pole of the substitution with matrix

$$\begin{vmatrix} 3 & 0 & -1 \\ 4 & -1 & -3 \\ 1 & 1 & 1 \end{vmatrix},$$

i.e. $3a - c = \lambda a,\ 4a - b - 3c = \lambda b,\ a + b + c = \lambda c.$

The only values satisfying these equations are

$$\lambda = 1,\ a : b : c = 1 : -1 : 2.$$

These give us $x - y + 2z = Ae^t$. Substitute for x in the original equations and we get

$$\dot{y} = -y + z,\ \dot{z} = -4y + 3z - Ae^t.$$

Multiply by b, c, and add, where $-b - 4c = \lambda b,\ b + 3c = \lambda c.$
These equations give $\lambda = 1,\ b : c = 2 : -1$;

$$\therefore\ 2y - z = (B + At)e^t.$$

This gives $\dot{y} = y - (B + At)e^t$ or $y = -(C + Bt + \frac{1}{2}At^2)e^t$, which combined with $x - y + 2z = Ae^t$, $2y - z = (B + At)e^t$ gives x, y, z.]

Ex. 3. Solve the equations

$$\left.\begin{aligned}\dot{x}&=3x-y\\ \dot{y}&=x+y\end{aligned}\right\},\quad \left.\begin{aligned}\dot{x}&=-5x+2y+e^{t}\\ \dot{y}&=x-6y+e^{2t}\end{aligned}\right\},\quad \left.\begin{aligned}t\dot{x}+2x-2y&=t\\ t\dot{y}+x+5y&=t^{2}\end{aligned}\right\}.$$

Ex. 4. Solve the equations

$$\left.\begin{aligned}\dot{x}&=y+z\\ \dot{y}&=z+x\\ \dot{z}&=x+y\end{aligned}\right\},\quad \left.\begin{aligned}\dot{x}&=-9x-8y+2z\\ \dot{y}&=8x+7y-2z\\ \dot{z}&=-8x-8y+z\end{aligned}\right\},\quad \left.\begin{aligned}\dot{x}+20x-13y+24z&=t\\ \dot{y}-15x+6y-16z&=2t\\ \dot{z}-24x-16y-29z&=3t\end{aligned}\right\},$$

$$\left.\begin{aligned}\dot{x}&=cy-bz\\ \dot{y}&=az-cx\\ \dot{z}&=bx-cy\end{aligned}\right\}.$$

Ex. 5. Solve the equation

$$(4x-3y-5z)\frac{\partial z}{\partial x}+(-x+2y+z)\frac{\partial z}{\partial y}=3x-3y-4z.$$

[This is a linear partial differential equation of Lagrange's type. The auxiliary equations are

$$\frac{dx}{4x-3y-5z}=\frac{dy}{-x+2y+z}=\frac{dz}{3x-3y-4z}$$
$$=\frac{a\,dx+b\,dy+c\,dz}{(4a-b+3c)x+(-3a+2b-3c)y+(-5a+b-4c)z}.$$

Choose a, b, c so that

$$4a-b+3c=\lambda a,\quad -3a+2b-3c=\lambda b,\quad -5a+b-4c=\lambda c,$$

i.e. so that (a, b, c) is a pole of the substitution with matrix

$$\begin{vmatrix}4&-1&3\\-3&2&-3\\-5&1&-4\end{vmatrix}.$$

These give $\lambda=1$ and $a:b:c=1:0:-1$,
or $\lambda=-1$ and $a:b:c=-1:1:2$,
or $\lambda=2$ and $a:b:c=-1:1:1$.

Hence the auxiliary equations may be written

$$\frac{d(x-z)}{(x-z)}=\frac{d(-x+y+2z)}{-(-x+y+2z)}=\frac{d(-x+y+z)}{2(-x+y+z)},$$

and the required solution is

$$(x-z)(-x+y+2z)=f\{(x-z)^2\div(-x+y+z)\}.]$$

Ex. 6. Solve the equations

$$\frac{\partial z}{\partial x}+(3y+4z)\frac{\partial z}{\partial y}=2y+5z,$$

$$-(y+z)\frac{\partial z}{\partial x}+(x+2y+z)\frac{\partial z}{\partial y}=3x+y+4z.$$

Ex. 7. Discuss the solution of

$$\ddot{x} = a_{i1}x_1 + a_{i2}x_2 + \dots + a_{im}x_m \ (i = 1, 2, \dots, m).$$

When will the solutions be periodic functions of t?

Ex. 8. Solve the equations

$$\left.\begin{aligned}\ddot{x}+m^2x&=0\\ \ddot{y}-m^2x&=0\end{aligned}\right\}, \quad \left.\begin{aligned}\ddot{x}+2x+2y&=\cos nt\\ \ddot{y}+\ x+3y&=\cos nt\end{aligned}\right\},$$

$$\ddot{y}+\ddot{z}+8x-7y+5z=\ddot{y}+8x-7y+6z=\ddot{x}-\ddot{y}-x+y=1.$$

Ex. 9. Solve the equations

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \dots,\ \dot{x}_{m-1} = x_m,\ \dot{x}_m = e_1x_1 + e_2x_2 + \dots + e_mx_m.$$

[We have $\dfrac{d^m x_1}{dt^m} = e_1x_1 + e_2\dfrac{dx_1}{dt} + \dots + e_m\dfrac{d^{m-1}x_1}{dt^{m-1}}$,

which gives x_1, and then $x_2, x_3, \dots, x_m$ are obtained by successive differentiation.]

Ex. 10. Use the last example to solve equations (i) of § 1.

[We suppose $f_1(t), f_2(t), \dots, f_m(t)$ zero—the method is similar in the case where these functions do not vanish.

Supposing $a_{12} \neq 0$,

we put $\quad a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1m}x_m = \xi_2,$

thus obtaining

$$\dot{x}_1 = \xi_2,\ \dot{\xi}_2 = p_{21}x_1 + p_{22}\xi_2 + p_{23}x_3 + \dots + p_{2m}x_m,\ \dots.$$

Supposing $p_{23} \neq 0$,

we put $\quad p_{21}x_1 + p_{22}\xi_2 + p_{23}x_3 + \dots + p_{2m}x_m = \xi_3,$

and so on, till we reduce equations (i) to the form

$$\dot{x}_1 = \xi_2,\ \dot{\xi}_2 = \xi_3,\ \dots,\ \dot{\xi}_{r-1} = \xi_r,\ \dot{\xi}_r = e_1x_1 + e_2\xi_2 + \dots + e_r\xi_r,$$

$$\dot{x}_{r+1} = b_{r+1\,1}x_1 + \dots + b_{r+1\,m}x_m,\ \dots,\ \dot{x}_m = b_{m1}x_1 + \dots + b_{mm}x_m.$$

The first r of these equations give $x_1, \xi_2, \dots, \xi_r$ in terms of t as in Ex. 9. Substitute these values in the remaining $m-r$ equations and repeat the process. As an alternative method we may reduce equations (i) to the sum of sets of equations of the type of Ex. 9 by the process of Ch. II, § 6.]

Ex. 11. Solve in this way

$$\dot{x} = x+y,\ \dot{y} = -x+y,\ \dot{z} = x+2y+z.$$

[Put $x+y=Y$.

Then $\quad \dot{x} = Y,\ \dot{Y} = -2x+2Y,\ \dot{z} = -x+2Y+z.$

The first two equations give $\ddot{x}-2\dot{x}+2x=0$, whence

$$x = e^t(A\cos t + B\sin t),\ x+y = Y$$
$$= e^t([A+B]\cos t + [-A+B]\sin t),$$

and then

$$\dot{z} - z = -x + 2Y = e^t([A+2B]\cos t + [-2A+B]\sin t)$$

gives $\quad z = e^t\{C + [A+2B]\sin t - [-2A+B]\cos t\}.$]

Ex. 12. Solve in this way any of the equations in Ex. 1, 2, 3, 4.

Ex. 13. Solve the equations

$$\dot{x}_1 = -k_1 x_1,\ \dot{x}_2 = -k_2 x_2 + k_1 x_1,\ \dot{x}_3 = -k_3 x_3 + k_2 x_2,\ \ldots,$$
$$\dot{x}_n = -k_n x_n + k_{n-1} x_{n-1},$$

where $$k_1 > k_2 > k_3 > \ldots > k_n,$$

and when $t = 0,\ x_1 = a,\ x_2 = x_3 = \ldots = x_n = 0.$

$$\left[x_2 = k_1 k_2 \ldots k_{n-1} a \times \sum_{i=1}^{i=n} \frac{e^{-k_i t}}{(k_1 - k_i)\ldots(k_{i-1}-k_i)(k_{i+1}-k_i)\ldots(k_n - k_i)}.\right]$$

Ex. 14. A substance A decomposes giving off a substance B, which decomposes giving off a substance C, which decomposes giving off a substance D, ... Given the time taken by any one of the substances to decompose to half its original bulk, find at any instant the amount of each substance present; assuming that originally a given quantity of A is alone present, and that the rate of decomposition of any substance is proportional to the amount of that substance in existence.

[We obtain equations similar to those of Ex. 13. For numerical examples see the Presidential Address of the British Association, Portsmouth, 1911.]

Ex. 15. Equal particles are arranged at equal distances in a row and are connected by light springs each at its natural length. Find the periods of longitudinal vibrations, and the principal coordinates when the two end particles are kept fixed.

[We have equations of the type

$$k\ddot{x}_1 = x_2 - 2x_1,\ k\ddot{x}_2 = x_3 - 2x_2 + x_1,\ \ldots,$$
$$k\ddot{x}_{n-1} = x_n - 2x_{n-1} + x_{n-2},\ k\ddot{x}_n = -2x_n + x_{n-1}.$$

The solution of these by § 1 involves finding the pole of a substitution similar to that discussed in Ch. I, § 6, Ex. 10.]

§ 2. The Solution of Simultaneous Linear Homogeneous Equations.

Suppose we have m linear homogeneous equations in m unknowns $x_1, x_2, \ldots, x_m$. We can put these equations into the form

$$\left.\begin{array}{ll} (a_{11}-\lambda)x_1 + a_{12}x_2 + \ldots + a_{1m}x_m & = 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \ldots + a_{2m}x_m & = 0 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + (a_{mm}-\lambda)x_m & = 0 \end{array}\right\} \ldots\ldots \text{(i)},$$

where λ is any constant chosen so that the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mm} \end{vmatrix}$$

does not vanish.

Then there is no solution of equations (i), excluding

$$x_1 = x_2 = \ldots = x_m = 0,$$

unless

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22}-\lambda & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mm}-\lambda \end{vmatrix} = 0.$$

If this relation is satisfied, a solution of (i) is

$$x_1 : x_2 : \ldots : x_m = X_1 : X_2 : \ldots : X_m,$$

where $(X_1, X_2, \ldots, X_m)$ is a pole of the substitution

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m \; (t = 1, 2, \ldots, m),$$

corresponding to the characteristic-root λ.

The number of such poles is given by Ch. II, § 5, Corollary V; it depends solely on the invariant-factors of the substitution which correspond to the characteristic-root λ.

In particular, if there is more than one value of

$$x_1 : x_2 : \ldots : x_m$$

satisfying (i), there is an infinite number of values.

Ex. 1. $$\left.\begin{aligned} x - y - z - 2w &= 0 \\ 2x + y + 4z - w &= 0 \\ x \quad + z - w &= 0 \\ x + 2y + 5z + w &= 0 \end{aligned}\right\}.$$

[The determinant

$$\begin{vmatrix} -\lambda & -1 & -1 & -2 \\ 2 & -\lambda & 4 & -1 \\ 1 & 0 & -\lambda & -1 \\ 1 & 2 & 5 & -\lambda \end{vmatrix}$$

is divisible by $(\lambda+1)^2$ and every first minor is divisible by $(\lambda+1)$.

Hence the substitution of which this is characteristic-determinant has a singly infinite number of poles corresponding to the characteristic-root $\lambda = -1$, so that the given equations have a doubly infinite number of solutions.]

Ex. 2. Discuss similarly

$$\left.\begin{aligned} 2x - y - 5z &= 0 \\ x + 3y + z &= 0 \\ 3x + 2y - 4z &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} x - y - z - w &= 0 \\ 3x - 4y - 4z - w &= 0 \\ 2x - 3y - 3z \phantom{{}-3w} &= 0 \\ 5x - 6y - 6z - 3w &= 0 \end{aligned}\right\}.$$

§ 3. Collineation.

If two figures are such that each point P of one figure corresponds to a single point P' of the other, while conversely P' corresponds to the single point P, one figure is said to be derived from the other by a *collinear*, or *projective*, or *homographic* transformation ('collineation'). First take the case in which both figures are *plane*.

If (x, y, z), (x', y', z') are the coordinates of P, P' referred to any two triangles of reference (one in each figure), we have evidently relations of the form

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z \\ y' &= l_2 x + m_2 y + n_2 z \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned}\right\} \quad \text{(i)}$$

If we choose the triangles of reference ABC, $A'B'C'$ so that A and A', B and B', C and C' are corresponding points in the two figures, $y' = z' = 0$ when $y = z = 0$, &c. Hence we have obviously $m_1 = n_1 = n_2 = l_2 = l_3 = m_3 = 0$. When we are given the coordinates of another pair of corresponding points we can find the ratios $l_1 : m_2 : n_3$. Hence a collinear transformation of one plane figure into another is completely determined by the correspondence between four points of one figure (no three of which are collinear) and four points of the other.

Suppose that in (i) we replace

x_1 by $a_1\xi + b_1\eta + c_1\zeta$, y by $a_2\xi + b_2\eta + c_2\zeta$, z by $a_3\xi + b_3\eta + c_3\zeta$ and x', y', z' by the same functions of ξ', η', ζ', and then solve for ξ', η', ζ', thus obtaining

$$\left.\begin{aligned} \xi' &= L_1\xi + M_1\eta + N_1\zeta \\ \eta' &= L_2\xi + M_2\eta + N_2\zeta \\ \zeta' &= L_3\xi + M_3\eta + N_3\zeta \end{aligned}\right\} \quad \text{(ii)}$$

Then (ii) gives the relation between the coordinates (ξ, η, ζ), (ξ', η', ζ') of P and P' referred to new triangles of reference.* But if we put x and x' for ξ and ξ', y and y' for η and η', z and z' for ζ and ζ' in (ii), we get by Ch. I, § 5, a transform of (i).

Hence a transformation of the fundamental substitution (i) defining the collineation is equivalent to a change of the triangles of reference.

Similarly, if the figures are three-dimensional, we can show by taking the vertices of the two tetrahedra of reference as corresponding points that the collinear transformation is completely determined by correspondence between five points of one figure (no four being coplanar) and five points of the other.

It is at once evident that to any number of coplanar points in one figure correspond coplanar points of the other. Similarly, to collinear points correspond collinear points, to concurrent lines correspond concurrent lines, to coaxial planes (through the same line) correspond coaxial planes, &c.

To four coaxial planes correspond four coaxial planes, forming a pencil of the same cross-ratio.

For let $u = 0$, $v = 0$ be the equations of two planes in one figure, and let $u' = 0$, $v' = 0$ be the equations of the corresponding planes in the other figure. Then to the planes

$$u = \lambda_1 v,\ u = \lambda_2 v,\ u = \lambda_3 v,\ u = \lambda_4 v$$

in one figure correspond the planes

$$u' = \lambda_1 v',\ u' = \lambda_2 v',\ u' = \lambda_3 v',\ u' = \lambda_4 v'$$

in the other. But the cross-ratio of both these pencils of planes is $\quad (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4) \div (\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2)$.

It follows at once that the cross-ratios of corresponding pencils of lines or ranges of points are identical.

Suppose now the planes whose collineation is defined by (i) to be superposed, and the same triangle of reference taken in each plane. Then P will coincide with its corresponding point P' if $x : y : z = x' : y' : z'$, i.e. if P is (X, Y, Z), where (X, Y, Z) is any pole of the substitution (i) (Ch. I, § 6).

The number of such 'self-corresponding' points depends on the invariant-factors of the substitution (i), as pointed out in Ch. II, § 5, Corollary V.

* For instance, the equations of the sides of the old triangle of reference referred to the new triangle of reference in the figure traced out by P are

$$a_1 x + b_1 y + c_1 z = 0,\ a_2 x + b_2 y + c_2 z = 0,\ a_3 x + b_3 y + c_3 z = 0.$$

Transformation of (i) into a 'canonical' substitution, which is equivalent to a suitable choice of a new triangle of reference, throws the connexion between the coordinates of P and P' into one of the three types

$$x' = \alpha x + y,\ y' = \alpha y + z,\ z' = \alpha z;$$
$$x' = \alpha x + y,\ y' = \alpha y,\qquad z' = \beta z;$$
$$x' = \alpha x,\qquad y' = \beta y,\qquad z' = \gamma z.*$$

If the collineation is of finite order, so that when we apply the collineation n times in succession to any figure we return to the original figure, the connexion between the coordinates can be put in the form

$$x' = \alpha x,\ y' = \beta y,\ z' = \gamma z,$$

where α, β, γ are roots of unity.

Similarly for three dimensions.

Ex. 1. A collineation of one straight line into another is completely determined when three pairs of corresponding points are given. If the lines coincide, there are two self-corresponding points.

[If x, x' are the distances of corresponding points from fixed origins on the lines, we have a relation of the form

$$pxx' + lx' + mx' + n = 0.$$

The ranges traced out by corresponding points are, of course, homographic (projective).]

Ex. 2. Find the self-corresponding points and the vanishing points (corresponding to the infinitely distant point on the line), when pairs of corresponding points in a collineation of a line into itself are at distances from the origin

(i) 3 and 1, -3 and $-\frac{1}{2}$, $\frac{1}{2}$ and -4,

(ii) 1 and -3, 2 and $-\frac{5}{2}$, -2 and $-\frac{3}{2}$,

(iii) 0 and -3, 1 and -1, 4 and 5.

[We have respectively

$$xx' - x' - 2 = 0,\ xx' + 2x + 1 = 0,\ -2x + x' + 3 = 0.]$$

Ex. 3. If two figures are superposable, one may be derived from the other by a collineation.

Find the self-corresponding points when one figure (in three dimensions) is derived from the other by rotation about an axis, a screw about an axis, or a translation.

Show also that two figures each of which is the reflexion of the other in a plane may be derived from each other by a collineation, and find the self-corresponding points.

* α, β, γ are not necessarily unequal.

Ex. 4. If a collineation of space into itself is of order 2, the segment joining corresponding points is divided harmonically by a fixed point and a fixed plane, or is divided harmonically by two fixed lines. What are the self-corresponding points in the two cases?

[The equations defining the collineation may be put into the form $$x' = x,\ y' = y,\ z' = z,\quad w' = -w,$$ or $$x' = x,\ y' = y,\ z' = -z,\ w' = -w.]$$

Ex. 5. What are the corresponding theorems for the collineation of a line or of a plane into itself?

Ex. 6. In a collineation of a plane into itself pairs of corresponding points have the homogeneous coordinates $(1, -1, 2)$ and $(0, 1, 4)$, $(0, 0, 1)$ and $(1, 1, -1)$, $(1, 1, 1)$ and $(-1, 3, 3)$, $(0, 1, 0)$, and $(1, 0, 2)$. Find the self-corresponding points.

[If (x_i', y_i', z_i') and (x_i, y_i, z_i) are a pair of corresponding points $(i = 1, 2, 3, 4)$, equations (i) give us
$$z_i'(l_1x_i + m_1y_i + n_1z_i) = x_i'(l_3x_i + m_3y_i + n_3z_i),$$
and seven other independent linear equations to find the eight ratios $l_1 : m_1 : n_1 : l_2 : m_2 : n_2 : l_3 : m_3 : n_3$, remembering that we only know the *ratios* $x_i' : y_i' : z_i'$ and $x_i : y_i : z_i$. In the case before us equations (i) become $x' = x - y - z$, $y' = 4x - z$, $z' = 4x - 2y + z$; whose real pole is $(1, 5, -6)$.]

Ex. 7. In a collineation of a plane into itself the points $(1, 0, 0)$ and $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$, $(0, 0, 1)$ and $(5, -1, 4)$, $(4, -2, 1)$ and $(1, -1, 0)$ correspond. Find the self-corresponding points.

[We have $x' = 2y + 5z$, $y' = -z$, $z' = -x + 4z$ with poles $(4, -1, 2)$ and $(3, -1, 1)$.]

Ex. 8. In a collineation of three-dimensional space into itself the points $(1, 0, 0, 0)$ and $(1, 0, -2, 0)$, $(0, 1, 0, 0)$ and $(-1, -3, 2, 2)$, $(0, 0, 1, 0)$ and $(2, 0, -3, 0)$, $(0, 0, 0, 1)$ and $(-2, -2, 3, 1)$, $(1, 1, 1, 1)$ and $(0, -5, 0, 3)$ correspond. Find the self-corresponding points.

We have
$$x' = x - y + 2z - 2w,\ y' = -3y - 2w,\ z' = -2x + 2y - 3z + 3w,$$
$$w' = 2y + w,$$
with poles $(X, -2, 1 - X, 2)$, so that any point on the line $y + w = 2x + y + 2z = 0$ is self-corresponding.]

Ex. 9. In a collineation of a plane into itself the points whose Cartesian coordinates are $(0, 0)$ and $(0, 0)$, $(0, 1)$ and $(0, 2)$, $(1, 0)$ and $(2, 1)$, $(1, 1)$ and $(3, 4)$ respectively correspond. Find the self-corresponding points and the vanishing lines of the plane.

[We may take z' and z unity in equations (i), so that a collineation of a plane in Cartesian coordinates is given by equations of the type
$$x'=\frac{l_1x+m_1y+n_1}{lx+my+n},\quad y'=\frac{l_2x+m_2y+n_2}{lx+my+n}.$$
In this case these equations become
$$x'=\frac{12x}{-x-2y+7},\qquad y'=\frac{6x+10y}{-x-2y+7},$$
$$\text{or } x=\frac{35x'}{-x'+12y'+60},\quad y=\frac{-21x'+42y'}{-x'+12y'+60}.$$
Therefore the self-corresponding points are $(0,\ 0)$, $(0,\ -\frac{3}{2})$, $(-\frac{5}{7},\ -\frac{15}{7})$, and the vanishing lines are $x+2y=7$, $x=12y+60$.]

Ex. 10. Discuss similarly the case where the corresponding pairs are $(0,\ 0)$ and $(0,\ 0)$, $(1,\ 1)$ and $(2,\ -1)$, $(0,\ 2)$ and $(-2,\ 4)$, $(-2,\ 0)$ and $(2,\ 2)$.
$$\left[x'=\frac{x+y}{x-y+1},\quad y'=\frac{x-2y}{x-y+1}.\right]$$

§ 4. Geometrical Movements.

Suppose that two real three-dimensional figures correspond to one another in a collineation, in such a way that the distance between any two points in one figure equals the distance between the corresponding points. Then one is said to be derived from the other by a 'geometrical movement'. Suppose that there exists a real finite self-corresponding point O. It may be taken as the origin of rectangular Cartesian coordinates. If $P\,(x,\ y,\ z)$, and $P'\,(x',\ y',\ z')$ are a pair of corresponding points, we have a relation (i) of § 3, which represents a real *orthogonal* substitution, since $OP^2=OP'^2$.

This relation can be transformed (Ch. I, § 15) into
$$x'=\cos\theta\,.\,x-\sin\theta\,.\,y,\ y'=\sin\theta\,.\,x+\cos\theta\,.\,y,\ z'=\pm z$$
by a real orthogonal substitution, which is equivalent evidently to a new choice of rectangular Cartesian axes having O as origin.

But this new relation informs us that P is brought to coincide with P' by a rotation through an angle θ about the axis of z, or by such a rotation followed by a reflexion in a plane through O perpendicular to this axis.

Hence :—

Any geometrical movement leaving a point O unmoved is a rotation about an axis through O, or such a rotation followed by reflexion in a plane through O perpendicular to this axis.

This result can readily be proved by geometry.

Ex. Any movement whatever is equivalent to a translation (without rotation) of one figure followed by a movement leaving one point unmoved.

Deduce that the movements in which there is no fixed finite point are a translation, a screw (translation parallel to an axis followed by rotation about that axis), and translation followed by reflexion in a plane parallel to the direction of translation.

Find the infinite fixed points in these cases.

§ 5. Solution of a Quartic Equation.*

The equation in t

$$a_0t^4+4a_1t^3+6a_2t^2+4a_3t+a_4=0$$

can be solved as follows:—

The left-hand side of the equation

$$a_0t^4+4a_1t^3+6a_2t^2+4a_3t+a_4$$

is equivalent to

$$a_0x^2+(a_2-a_0\theta)y^2+a_4z^2+2a_3yz+2(a_2+2a_0\theta)zx+2a_1xy,$$

where $\quad x=t^2,\ y=2t,\ z=1.$

Now this last expression can be expressed as the sum of *two* squares (Ch. III, § 6), and can therefore be at once factorized, provided

$$\begin{vmatrix} a_0 & a_1 & a_2+2a_0\theta \\ a_1 & a_2-a_0\theta & a_3 \\ a_2+2a_0\theta & a_3 & a_4 \end{vmatrix}=0,$$

which is the well-known 'reducing cubic' of the given equation †

Ex. 1. If θ is any root of the 'reducing cubic', prove that

$$\begin{aligned} a_0(a_0t^4+4a_1t^3+6a_2t^2+4a_3t+a_4) &\equiv (a_0t^2+2a_1t+a_2+2a_0\theta)^2 \\ &\qquad -(2t\sqrt{a_1^2-a_0a_2+a_0^2\theta}+\sqrt{(a_2+2a_0\theta)^2-a_0a_4})^2 \\ &\equiv \{a_0t^2+2(a_1+\sqrt{a_1^2-a_0a_2+a_0^2\theta})t+\dots\} \\ &\qquad \{a_0t^2+2(a_1-\sqrt{a_1^2-a_0a_2+a_0^2\theta})t+\dots\}. \end{aligned}$$

Ex. 2. Show that, if $\alpha, \beta, \gamma, \delta$ are the roots of the quartic, and $\theta_1, \theta_2, \theta_3$ the roots of the reducing cubic,

$$a_0(\alpha-\beta-\gamma+\delta)=4\sqrt{a_1^2-a_0a_2+a_0^2\theta_1},\ \&c.,$$

* See Heilermann, *Zeitschr. Math. Phys.*, xliv (1898), p. 234.

† See Burnside and Panton's *Theory of Equations*, Ch. VI.

and hence that

$$a_0\alpha = (-a_1+e_1-e_2-e_3),\quad a_0\beta = (-a_1-e_1+e_2-e_3),$$
$$a_0\gamma = (-a_1-e_1-e_2+e_3),\quad a_0\delta = (-a_1+e_1+e_2+e_3),$$

where $$e_1^2 = a_1^2-a_0a_2+a_0^2\theta_1, \text{ \&c.},$$

and $$2e_1e_2e_3 = 3a_0a_1a_2-2a_1^3-a_0^2a_3.$$

Ex. 3. Prove that $12\theta_1 = 2(\beta\gamma+\alpha\delta)-(\gamma\alpha+\beta\delta)-(\alpha\beta+\gamma\delta)$, &c.

Ex. 4. Solve the equation $t^4+12t+3=0$.

[The reducing cubic is $4\theta^3-3\theta-9=0$, one of whose roots is $\theta=\frac{3}{2}$. The given equation becomes

$$0 = x^2-\tfrac{3}{2}y^2+3z^2+6yz-6zx = (x+3z)^2-\tfrac{3}{2}(y-2z)^2,$$

where $$x=t^2,\ y=2t,\ z=1.$$

Therefore the given equation is

$$(t^2-\surd 6t+3+\surd 6)(t^2+\surd 6t+3-\surd 6)=0.]$$

Ex. 5. Solve
$$\begin{aligned} t^4+\ 6t^3+12t^2+14t+3 &= 0,\\ t^4+12t-5 &= 0,\\ 4t^4+24t^3+24t^2-28t+3 &= 0,\\ t^4+\ 6t^2-\ 4t+24 &= 0. \end{aligned}$$

§ 6. Critical Values of $\Sigma\, a_{ij}x_ix_j$ $(i, j = 1, 2, \ldots, m)$.

If the determinant of the real quadratic form

$$\Sigma\, a_{ij}x_ix_j\ (a_{ij}=a_{ji})$$

is of rank r, we can find by Ch. III, § 2, real independent linear functions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ of $x_1, x_2, \ldots, x_m$ such that

$$\Sigma\, a_{ij}x_ix_j \equiv \lambda_1\mathbf{x}_1^2+\lambda_2\mathbf{x}_2^2+\ldots+\lambda_r\mathbf{x}_r^2,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_r$ are the (real) non-zero roots of

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdot & \cdot & \cdot & a_{1m} \\ a_{21} & a_{22}-\lambda & \cdot & \cdot & \cdot & a_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mm}-\lambda \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now $S\equiv\lambda_1\mathbf{x}_1^2+\lambda_2\mathbf{x}_2^2+\ldots+\lambda_r\mathbf{x}_r^2$ cannot be critical unless its first partial derivatives with respect to $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ all vanish; i.e. unless $\mathbf{x}_1=\mathbf{x}_2=\ldots=\mathbf{x}_r=0$. It will really be critical in this case, if and only if $\lambda_1, \lambda_2, \ldots, \lambda_r$ have the same sign. For if $\lambda_1>0$, $\lambda_2<0$, S is positive when

$$\mathbf{x}_2=\mathbf{x}_3=\ldots=\mathbf{x}_r=0,$$

and negative when $\mathbf{x}_1=\mathbf{x}_3=\ldots=\mathbf{x}_r=0$.

Hence $\Sigma\, a_{ij}x_ix_j$ has a maximum value if $\lambda_1, \lambda_2, \ldots, \lambda_r$ are

all negative, and a minimum value if $\lambda_1, \lambda_2, \ldots, \lambda_r$ are all positive; but has no critical (maximum or minimum) value in any other case.

Let $f(x_1, x_2, \ldots, x_m)$ be any function of $x_1, x_2, \ldots, x_m$, such that
$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \ldots = \frac{\partial f}{\partial x_m} = 0,$$
when
$$x_1 = X_1,\ x_2 = X_2,\ \ldots,\ x_m = X_m.$$

Put $x_1 = X_1 + \xi_1,\ x_2 = X_2 + \xi_2.\ \ldots,\ x_m = X_m + \xi_m.$

Then if
$$f(x_1, x_2, \ldots, x_m) - f(X_1, X_2, \ldots, X_m) = \Sigma\, a_{ij}\xi_i\xi_j + R,$$
where R contains terms of the 3rd, 4th, ... degrees in $\xi_1, \xi_2, \ldots, \xi_m$,* $f(x_1, x_2, \ldots, x_m)$ will be critical or not critical when
$$x_1 = X_1,\ x_2 = X_2,\ \ldots,\ x_m = X_m$$
according as the non-zero roots of (i) have or have not all the same sign; provided we may assume that $\Sigma\, a_{ij}\xi_i\xi_j + R$ has the same sign as $\Sigma\, a_{ij}\xi_i\xi_j$ for all values of $\xi_1, \xi_2, \ldots, \xi_m$ numerically less than a given quantity ξ.

The assumption is not *always* justified, as may be seen by taking the simple example
$$f \equiv 3x_1^4 - 4x_1^2 x_2 + x_2^2,\ X_1 = X_2 = 0,\ \xi_2 = 2\xi_1^2,$$
when
$$\Sigma\, a_{ij}\xi_i\xi_j \equiv 4\xi_1^4 \text{ and } \Sigma\, a_{ij}\xi_i\xi_j + R = -\xi_1^4.$$

The following method † is often of practical value:—

Suppose $f(x, y)$ is *any* polynomial in x and y. We can find its critical values as follows. Suppose $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ when $x = a,\ y = b$.

Then $f(x, y) - f(a, b) = 0$ is the equation of a curve with a real double point at (a, b).

If a *real* branch of the curve goes through (a, b), then $f(x, y) - f(a, b)$ changes sign whenever the point (x, y) crosses this branch as it moves in any manner near the fixed point (a, b). Hence $f(x, y)$ is critical when $x = a,\ y = b$, if and only if $f(x, y) - f(a, b) = 0$ has no real branch passing through the real double point (a, b).

To find whether such a real branch exists, transfer the origin to (a, b) and use Newton's diagram.

Similarly we may discuss the critical value of any polynomial $f(x, y, z)$ in three variables x, y, z.

* $a_{ij} = \dfrac{\partial^2 f}{\partial X_i \partial X_j}$ in general.

† Bromwich, *Quadratic Invariants*, § 9

Ex. 1. $x^2+3y^2+4z^2+2yz+2zx-2xy$ is a minimum when $x=y=z=0$.

[
$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 3-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = -\lambda^3+8\lambda^2-16\lambda+2\,;$$

and the roots of $\lambda^3-8\lambda^2+16\lambda-2=0$, which are certainly real, are evidently all positive

Another method of proof is to express the given quantity in the form $(x-y+z)^2+2(y+z)^2+z^2$ as in Ch. III, § 6. It is then seen to be a minimum, as stated above.]

Ex. 2. Discuss

$$x^2+2xy+2y^2,\ x^2+z^2-2yz+2zx+2xy,$$
$$x^2+5y^2+4z^2-2yz-2zx+4xy,$$
$$x^2+2y^2+6z^2+12w^2+2yz-2zx+2xy-2xw,$$
$$yz+zx+xy+xw+yw+zw.$$

Ex. 3. Find the critical values of

$$f(x,\,y)\equiv 3x^6+6x^3y-2y^3.$$

[$\dfrac{\partial f}{\partial x}=\dfrac{\partial f}{\partial y}=0$ when $x=y=0$, or $x=1$, $y=-1$.

$f(x,y)=f(0,0)$ has real branches through $(0,0)$; but

$$f(x,\,y)=f(1,\,-1)$$

has an isolated point at $(1,\,-1)$, as is seen by transferring the origin to this point.

Hence $f(x,\,y)$ is not critical when $x=y=0$, but is a minimum when $x=1$, $y=-1$.]

Ex. 4. Find the critical values of

$$7y^2+14x^3y+6x^7,\ x^3+y^3-3axy,\ x(x^2+y^2)-3axy,$$
$$2x^5+y^5-5x^2y,\ x^4+y^4-2x^2+4xy-2y^2,$$
$$x^2y^2+6c^3(3x+2y),\ xyz+(y+z)^2+(x+2)^2,$$
$$y^2+2z^2-5x^4+4x^5.$$

§ 7. The Classification of Conicoids.

Suppose

$$S\equiv ax^2+by^2+cz^2+dw^2+2fyz+2gzx$$
$$+2hxy+2lxw+2myw+2nzw=0$$

is the real equation of a conicoid, x, y, z, w being homogeneous coordinates.

By a real (orthogonal) change of variables, S may be reduced to the form (Ch. III, § 2)

$$\mathbf{S}\equiv\lambda_1\mathbf{x}^2+\lambda_2\mathbf{y}^2+\lambda_3\mathbf{z}^2+\lambda_4\mathbf{w}^2,$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the real roots of

$$\begin{vmatrix} a-\lambda & h & g & l \\ h & b-\lambda & f & m \\ g & f & c-\lambda & n \\ l & m & n & d-\lambda \end{vmatrix} = 0 \text{............(i)}$$

Geometrically this is equivalent to taking a self-conjugate tetrahedron as a new tetrahedron of reference.

The nature of the conicoid will be determined by a knowledge of the values $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

First suppose none of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ zero; i.e.

$$\Delta \equiv \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}$$

is not zero (is of rank 4).

We have then three possibilities:—

(1) $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ have all the same sign. Then $S = 0$ is an imaginary conicoid; for $\mathbf{s}$ (and therefore S) is always one-signed for real values of x, y, z, w.

(2) Three of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are positive and the fourth is negative, or vice versa.

Then $S = 0$ is a real conicoid with unreal generators; for the generators of $\mathbf{s} = 0$ are

$$\left.\begin{aligned} [\lambda_1^{\frac{1}{2}}\mathbf{x} + (-\lambda_2)^{\frac{1}{2}}\mathbf{y}] &= k\,[(-\lambda_3)^{\frac{1}{2}}\mathbf{z} + \lambda_4^{\frac{1}{2}}\mathbf{w}] \\ k\,[\lambda_1^{\frac{1}{2}}\mathbf{x} - (-\lambda_2)^{\frac{1}{2}}\mathbf{y}] &= [(-\lambda_3)^{\frac{1}{2}}\mathbf{z} - \lambda_4^{\frac{1}{2}}\mathbf{w}] \end{aligned}\right\}$$

and

$$\left.\begin{aligned} [\lambda_1^{\frac{1}{2}}\mathbf{x} + (-\lambda_2)^{\frac{1}{2}}\mathbf{y}] &= k\,[(-\lambda_3)^{\frac{1}{2}}\mathbf{z} - \lambda_4^{\frac{1}{2}}\mathbf{w}] \\ k\,[\lambda_1^{\frac{1}{2}}\mathbf{x} - (-\lambda_2)^{\frac{1}{2}}\mathbf{y}] &= [(-\lambda_3)^{\frac{1}{2}}\mathbf{z} + \lambda_4^{\frac{1}{2}}\mathbf{w}] \end{aligned}\right\}.$$

(3) Two of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are positive and the other two are negative.

Then $S = 0$ is a real conicoid with real generators.

Next suppose one of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ (say λ_4) is zero; so that Δ is of rank 3 since the determinant of $\mathbf{s}$ is evidently of rank 3 (Ch. III, § 6).

Then $S = 0$ is a cone with vertex at $\mathbf{x} = \mathbf{y} = \mathbf{z} = 0$. The cone is imaginary if $\lambda_1, \lambda_2, \lambda_3$ have the same sign; and is real if two of $\lambda_1, \lambda_2, \lambda_3$ are positive and the other negative, or vice versa.

Next suppose two of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ (say λ_3 and λ_4) are zero, so that Δ is of rank 2.

Then $S = 0$ is a pair of planes meeting in the real line $\mathbf{x} = \mathbf{y} = 0$. The pair is imaginary if λ_1 and λ_2 have the same sign, and is real if λ_1 and λ_2 have opposite signs.

Lastly, suppose three of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are zero, so that Δ is of rank 1.

Then $S = 0$ is a pair of coincident planes.

Now let $S = 0$ be the Cartesian equation obtained by putting $w = 1$.

Let μ_1, μ_2, μ_3 be the real roots of

$$\begin{vmatrix} a-\mu & h & g \\ h & b-\mu & f \\ g & f & c-\mu \end{vmatrix} = 0. \qquad \text{(ii)}$$

The consideration of the nature of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

joining the origin to the intersection of the conicoid with the plane at infinity $w = 0$, leads at once to the following results :—

First let $\Delta \neq 0$.

If

$$D \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0,$$

the conicoid is an ellipsoid when μ_1, μ_2, μ_3 have the same sign, and a hyperboloid when two of μ_1, μ_2, μ_3 have one sign and the other has the opposite sign.* If $\mu_3 = 0$ (D of rank 2), the conicoid is an elliptic paraboloid if μ_1 and μ_2 have the same sign,† and is a hyperbolic paraboloid if μ_1 and μ_2 have opposite signs.

Next suppose Δ of rank 3.

If $D \neq 0$, the conicoid is a cone.‡

If $\mu_3 = 0$ (D of rank 2), the conicoid is an elliptic cylinder when μ_1 and μ_2 have the same sign,§ and a hyperbolic cylinder when μ_1 and μ_2 have opposite signs.

If $\mu_2 = \mu_3 = 0$ (D of rank 1), the conicoid is a parabolic cylinder.

Next suppose Δ of rank 2.

If $\mu_3 = 0$ (D of rank 2), the conicoid is a pair of planes.

If $\mu_2 = \mu_3 = 0$ (D of rank 1), the conicoid is a pair of parallel planes.

* Since $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ is reduced to $\mu_1 \mathbf{x}^2 + \mu_2 \mathbf{y}^2 + \mu_3 \mathbf{z}^2$ by a real *orthogonal* substitution, the conicoid will be one of revolution if two of μ_1, μ_2, μ_3 are equal (and a sphere if all three are equal) on the supposition that the Cartesian axes of reference are rectangular.

† A paraboloid of revolution if $\mu_1 = \mu_2$.

‡ A right circular cone if two of μ_1, μ_2, μ_3 are equal.

§ A right circular cylinder if $\mu_1 = \mu_2$.

It should be noticed that we are only concerned with the *signs* of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. If we reduce S to the form

$$\Lambda_1 \xi^2 + \Lambda_2 \eta^2 + \Lambda_3 \zeta^2 + \Lambda_4 \omega^2$$

by any real transformation (for instance, by the method of Ch. III, § 6), it will follow from Ch. III, § 2, that the signs of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the same as those of $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$.

Similarly for the reduction of

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

to the sum (or difference) of squares.

Ex. 1. Find the nature of the conicoid

$$x^2 + y^2 + z^2 + 2yz - 2x - 8y - 3 = 0.$$

[Equation (i) is $\lambda^4 - 24\lambda^2 + 40\lambda - 16 = 0$; three of the roots of this equation are positive and one negative, so that the conicoid has no real generator.

Equation (ii) is $\mu(\mu-1)(\mu-2) = 0$; one root of this equation is zero and the other two positive, so that the conicoid has two imaginary generators at infinity.

Hence the conicoid is an elliptic paraboloid.

Otherwise: write the conicoid as

$$(x-1)^2 + (y+z)^2 + 4y^2 - 4(y+1)^2 = 0,$$

and the terms of the second degree as $x^2 + (y+z)^2$.]

Ex. 2. Discuss similarly

$$\begin{aligned}
&x^2 + 4y^2 - z^2 - 2yz - zx + 4xy + 2z = 0,\\
&x^2 + 9y^2 - 6xy + 2y - 4z = 0,\\
&-4x^2 + 9y^2 - z^2 + 3 + 4zx + 4x - 12y - 2z = 0,\\
&x^2 - 4y^2 - 3z^2 + 8yz + 2zx + 2x = 0,\\
&5(x^2 + y^2 + z^2) - 8yz + 6xy + 2z = 0,\\
&2x^2 + 3y^2 + z^2 - 2yz - 4xy + 4x - 4y + 2 = 0,\\
&2x^2 + 5y^2 + z^2 - 4xy - 2x - 4y - 8 = 0.
\end{aligned}$$

[Hyperbolic cylinder, parabolic cylinder, real plane-pair, hyperbolic paraboloid, elliptic paraboloid, unreal plane-pair, ellipsoid of revolution.]

Ex. 3. Discuss similarly

$$\begin{aligned}
&z^2 + 2nz + 2fyz + 2gzx + 2hxy = 0,\\
&x^2 + y^2 + cz^2 + 2gzx + 2nz = 0,\\
&2yz + 2zx + 2xy - 2x - 4y - 6z + d = 0.
\end{aligned}$$

Ex. 4. For what value of k is

$$x^2 + 5y^2 - 2z^2 + kw^2 - 6yz + 2zx - 4xy + 2xw - 6yw + 8zx = 0$$

(i) a cone, (ii) a conicoid with real generators, (iii) a conicoid with unreal generators.

[$k = 1,\ k < 1,\ k > 1$.]

Ex. 5. A real conicoid touching the edges of the tetrahedron of reference can be put in the form

$$a^2x^2+b^2y^2+c^2z^2+d^2w^2-2bcyz-2cazx-2abxy-2adxw -2bdyw-2cdzw = 0,$$

and it has no real generator.

[Equation (i) has $-16a^2b^2c^2d^2$ as the product of its roots.]

Ex. 6. Discuss

$$a^2x^2+b^2y^2+c^2z^2+d^2w^2-2bcyz-2cazx-2abxy-2adxw -2bdyw+2cdzw = 0,$$

$$mnyz+nlzx+lmxy+lpxw+mpyw+npzw = 0.$$

[Cone; no real generator.]

Ex. 7. Find the asymptotes and centre of the conic

$$x^2-3xy+2y^2+3x-4y = 0.$$

[Applying the method of Ch. III, § 6, we get

$$x^2-3xy+2y^2+3x-4y = (x-\tfrac{3}{2}y+\tfrac{3}{2})^2-(\tfrac{1}{2}y-\tfrac{1}{2})^2-2.$$

Hence the asymptotes are $(x-\frac{3}{2}y+\frac{3}{2})^2-(\frac{1}{2}y-\frac{1}{2})^2 = 0$, and the centre is $(x-\frac{3}{2}y+\frac{3}{2}) = (\frac{1}{2}y-\frac{1}{2}) = 0$; i.e. the asymptotes are $(x-y+1)(x-2y+2) = 0$ and the centre is $(0, 1)$.]

Ex. 8. Find by the method of Ex. 7 the asymptotes and centre of

$$3x^2-5xy-2y^2+9x-4y-12 = 0,$$
$$x^2-xy-2y^2+3x+8 = 0,$$
$$4x^2+24xy+11y^2+18x-11y-2 = 0,$$
$$5x^2-4xy+8y^2-20x+8y-16 = 0.$$

Ex. 9. Find the asymptotic cone and centre of the conicoid

$$x^2+y^2-z^2+2yz+2zx-2xy+2x-6y+2z+2 = 0.$$

[Applying the method of Ch. III, § 6, we get

$$x^2+y^2-z^2+2yz+2zx-2xy+2x-6y+2z+2 = (x-y+z+1)^2-2(y-z)^2+2(y-1)^2-1.$$

Hence the asymptotic cone is

$$(x-y+z+1)^2-2(y-z)^2+2(y-1)^2 = 0,$$

and the centre is

$$(x-y+z+1) = (y-z) = (y-1) = 0, \text{ i.e. is } (-1, 1, 1).]$$

Ex. 10. Find by the method of Ex. 9 the asymptotic cone and centre of

$$7x^2+6y^2+5z^2-4yz-4xy-14x+12y-10z+21 = 0,$$
$$x^2+z^2+2zx-2xy-4x+2z-1 = 0,$$
$$x^2+2y^2+6z^2-2yz+4zx-2xy-2x+4y+2z+1 = 0.$$

§ 8. The Tangential Equation of a Conicoid.

The condition that the conicoid S of § 7 should touch the plane $\lambda x+\mu y+\nu z+\pi w=0$, i.e. the tangential equation of the conicoid, is $\Sigma(\lambda,\mu,\nu,\pi)=0$, where

$$\Sigma(x,y,z,w)\equiv Ax^2+By^2+Cz^2+Dw^2+2Fyz+2Gzx$$
$$+2Hxy+2Lxw+2Myw+2Nzw$$

is the quadratic form *adjoint* to S (Ch. III, § 8).

If the conicoid reduces to a pair of planes, the determinant Δ of § 7 is of rank 1 or 2, and hence $\Sigma(\lambda,\mu,\nu,\pi)$ vanishes identically.

If the conicoid reduces to a cone, Δ is of rank 3. Hence $\Sigma(\lambda,\mu,\nu,\pi)$ is a square, and $\Sigma=0$ represents a pair of coincident points.

This pair is the vertex (X, Y, Z, W) of the cone. For since the polar plane of any point with respect to a cone passes through the vertex of the cone, $x=X$, $y=Y$, $z=Z$, $w=W$ is a solution of

$$\frac{\partial S}{\partial x}=\frac{\partial S}{\partial y}=\frac{\partial S}{\partial z}=\frac{\partial S}{\partial w}=0.$$

Hence by Ch. III, § 8,

$$\Sigma(\lambda,\mu,\nu,\pi)\equiv(\lambda X+\mu Y+\nu Z+\pi W)^2.$$

Ex. 1. The tangential equation of a conic which is a line-pair represents the intersection of the pair twice over; unless the two lines coincide, when the tangential equation is an identity.

Ex. 2. Interpret the results of § 8 and Ex. 1 geometrically.

[For instance, any plane touching a cone passes through the vertex; so that the equation $\Sigma=0$ cannot represent anything but the vertex. But Σ is of the second degree in λ, μ, ν, π, and therefore $\Sigma=0$ must be the vertex twice over.]

Ex. 3. If we change the equation of a surface by operating with a substitution A on the variables x, y, z, w, we obtain the new tangential equation of the surface by operating with A'^{-1} on λ, μ, ν, π.

§ 9. Relations between Two Conics.

It was pointed out in Ch. III, § 1, that if we transform the quadratic form

$$\sum_{i,j}(a_{ij}-\lambda b_{ij})x_ix_j,\ \text{ where } a_{ij}=a_{ji} \text{ and } b_{ij}=b_{ji},$$

into

$$\sum_{i,j}(c_{ij}-\lambda d_{ij})\mathbf{x}_i\mathbf{x}_j$$

by suitable change of variables, the determinants of the original and transformed quadratic forms have the same invariant-factors.

Take now the case of three variables x, y, z; and let

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$
$$S' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy.$$

If the conics $S=0$, $S'=0$ have the relation shown in the third column of the appended table, it is possible by a suitable change of variables (i.e. a suitable choice of triangle of reference) to transform S into σ and S' into σ', where σ and σ' are given in the second column.

The invariant-factors of the determinant of $\sigma - \lambda\sigma'$ given in the first column of the table are obvious by inspection, and they are the same as those of the determinant of $S - \lambda S'$.

Since column 1 gives every possible combination of invariant-factors, and column 3 every possible relation (of the kind considered) between the conics when S' is non-degenerate, we may conclude conversely that a knowledge of the invariant-factors of the determinant of $S - \lambda S'$ gives us the relation between the conics.

Invariant-factors.	Simplified equations.	Relation between conics.
$(\lambda-\alpha)$ $(\lambda-\beta)$ $(\lambda-\gamma)$	$\sigma \equiv \alpha x^2 + \beta y^2 + \gamma z^2$ $\sigma' \equiv x^2 + y^2 + z^2$	No special relation.
$(\lambda-\alpha)$ $(\lambda-\alpha)$ $(\lambda-\beta)$	$\sigma \equiv \alpha x^2 + \alpha y^2 + \beta z^2$ $\sigma' \equiv x^2 + y^2 + z^2$	Conics have double contact.
$(\lambda-\alpha)$ $(\lambda-\alpha)$ $(\lambda-\alpha)$	$\sigma \equiv \alpha x^2 + \alpha y^2 + \alpha z^2$ $\sigma' \equiv x^2 + y^2 + z^2$	Conics coincide.
$(\lambda-\alpha)^2$ $(\lambda-\beta)$	$\sigma \equiv 2\alpha xy + y^2 + \beta z^2$ $\sigma' \equiv 2xy + z^2$	Conics have single contact.
$(\lambda-\alpha)^2$ $(\lambda-\alpha)$	$\sigma \equiv 2\alpha xy + y^2 + \alpha z^2$ $\sigma' \equiv 2xy + z^2$	Conics have four-point contact.
$(\lambda-\alpha)^3$	$\sigma \equiv \alpha(2xz + y^2) + 2yz$ $\sigma' \equiv 2xz + y^2$	Conics have three-point contact.

A similar method can be applied to two conicoids.*

* See also Ch. IX, § 3, Ex. 4.

For the sake of completeness a table of the results is given, adapted from Bromwich's *Quadratic Invariants*, p. 46.

Invariant-factors.	Simplified equations.	Relation between conicoids.
$(\lambda-\alpha)$ $(\lambda-\beta)$ $(\lambda-\gamma)$ $(\lambda-\delta)$	$\sigma \equiv \alpha x^2+\beta y^2+\gamma z^2+\delta w^2$ $\sigma' \equiv x^2+y^2+z^2+w^2$	No special relation.
$(\lambda-\alpha)$ $(\lambda-\alpha)$ $(\lambda-\beta)$ $(\lambda-\gamma)$	$\sigma \equiv \alpha x^2+\alpha y^2+\beta z^2+\gamma w^2$ $\sigma' \equiv x^2+y^2+z^2+w^2$	Conicoids meet in two conics.
$(\lambda-\alpha)$ $(\lambda-\alpha)$ $(\lambda-\beta)$ $(\lambda-\beta)$	$\sigma \equiv \alpha x^2+\alpha y^2+\beta z^2+\beta w^2$ $\sigma' \equiv x^2+y^2+z^2+w^2$	Conicoids have four generators in common.
$(\lambda-\alpha)$ $(\lambda-\alpha)$ $(\lambda-\alpha)$ $(\lambda-\beta)$	$\sigma \equiv \alpha x^2+\alpha y^2+\alpha z^2+\beta w^2$ $\sigma' \equiv x^2+y^2+z^2+w^2$	Conicoids have ring-contact.
$(\lambda-\alpha)$ $(\lambda-\alpha)$ $(\lambda-\alpha)$ $(\lambda-\alpha)$	$\sigma \equiv \alpha x^2+\alpha y^2+\alpha z^2+\alpha w^2$ $\sigma' \equiv x^2+y^2+z^2+w^2$	Conicoids coincide.
$(\lambda-\alpha)^2$ $(\lambda-\beta)$ $(\lambda-\gamma)$	$\sigma \equiv 2\alpha xy+y^2+\beta z^2+\gamma w^2$ $\sigma' \equiv 2xy+z^2+w^2$	Conicoids touch at one point.
$(\lambda-\alpha)^2$ $(\lambda-\alpha)$ $(\lambda-\beta)$	$\sigma \equiv 2\alpha xy+y^2+\alpha z^2+\beta w^2$ $\sigma' \equiv 2xy+z^2+w^2$	Conicoids meet in two conics which touch.
$(\lambda-\alpha)^2$ $(\lambda-\beta)$ $(\lambda-\beta)$	$\sigma \equiv 2\alpha xy+y^2+\beta z^2+\beta w^2$ $\sigma' \equiv 2xy+z^2+w^2$	Conicoids meet in a conic and two generators intersecting on the conic.
$(\lambda-\alpha)^2$ $(\lambda-\alpha)$ $(\lambda-\alpha)$	$\sigma \equiv 2\alpha xy+y^2+\alpha z^2+\alpha w^2$ $\sigma' \equiv 2xy+z^2+w^2$	Conicoids touch along two generators.
$(\lambda-\alpha)^2$ $(\lambda-\beta)^2$	$\sigma \equiv 2\alpha xy+y^2+2\beta zw+w^2$ $\sigma' \equiv 2xy+2zw$	Conicoids meet in a generator and a cubic.

Invariant-factors.	Simplified equations.	Relation between conicoids.
$(\lambda-\alpha)^2$ $(\lambda-\alpha)^2$	$\sigma\equiv 2\alpha xy+y^2+2\alpha zw+w^2$ $\sigma'\equiv 2xy+2zw$	Conicoids meet in four generators, two of which coincide.
$(\lambda-\alpha)^3$ $(\lambda-\beta)$	$\sigma\equiv\alpha(2xz+y^2)+2yz+\beta w^2$ $\sigma'\equiv 2xz+y^2+w^2$	Conicoids have stationary contact at one point.
$(\lambda-\alpha)^3$ $(\lambda-\alpha)$	$\sigma\equiv\alpha(2xz+y^2)+2yz+\alpha w^2$ $\sigma'\equiv 2xz+y^2+w^2$	Conicoids meet in a conic and two generators which meet on the conic and whose plane touches the conic.
$(\lambda-\alpha)^4$	$\sigma\equiv\alpha(2xw+2yz)+2yw+z^2$ $\sigma'\equiv 2xw+2yz$	Conicoids meet in a generator and a cubic touching the generator.

Ex. 1. $S\equiv x^2-z^2+2zx+2yz=0,\ S'\equiv x^2+3z^2+2zx-2yz=0.$

[The determinant of $S-\lambda S'$ is

$$\begin{vmatrix} 1-\lambda & 0 & 1-\lambda \\ 0 & 0 & 1+\lambda \\ 1-\lambda & 1+\lambda & -1-3\lambda \end{vmatrix}\equiv(\lambda-1)(\lambda+1)^2.$$

$\lambda+1$ is a factor of each first minor; therefore the invariant-factors are $\lambda+1,\ \lambda+1,\ \lambda-1$, and the conics have double contact.]

Ex. 2. $S\equiv y^2+2yz-z^2+2xz=0,\ S'\equiv y^2+2xz=0.$

[The determinant of $S-\lambda S'$ is

$$\begin{vmatrix} 0 & 0 & 1-\lambda \\ 0 & 1-\lambda & 1 \\ 1-\lambda & 1 & -1 \end{vmatrix}=(\lambda-1)^3.$$

$\lambda-1$ is not a factor of each first minor. The invariant-factors are $(\lambda-1)^3$; and the conics have three-point contact.]

Ex. 3.

$S\equiv 6x^2-y^2+2z^2+6yz+4xy=0,\quad S'\equiv 3x^2-y^2-z^2-6zx-2xy=0.$

$S\equiv x^2+2zx-yz=0,\quad S'\equiv x^2-3z^2+2zx-4yz=0.$

$S\equiv y^2-4xz-8z^2=0,\quad S'\equiv y^2+4yz+4z^2+2xz=0.$

$S\equiv x^2+y^2-z^2=0,\quad S'\equiv 2x^2+y^2+2xz=0.$

$S\equiv x^2+y^2-yz-zx=0,\quad S'\equiv x^2+3y^2+z^2-4yz-2zx=0.$

[No contact, double contact, simple contact, four-point contact, three-point contact.]

Ex. 4.

$$\left.\begin{aligned} S &\equiv 3y^2+7z^2+15w^2+10yz-4zx+4xy-4xw-14yw-18zw=0 \\ S' &\equiv 5x^2+2y^2+3z^2+5w^2+4zx-4xy+4xw-6yw-2zw=0 \end{aligned}\right\}.$$

$$S \equiv yz-xw=0, \qquad S'=yz+xw=0.$$

$$\left.\begin{aligned} S &\equiv -4x^2+3w^2-4yz-8zx-4xw-2yw-12zw=0 \\ S' &\equiv 4x^2+y^2+5z^2+2w^2+4yz+8zx+6xy+4xw+2yw+2zw=0 \end{aligned}\right\}.$$

$$S \equiv x^2-3xy+2y^2+2zw=0, \qquad S' \equiv 2x^2+xy-3y^2+2zw=0.$$

[Ring-contact, four generators in common, contact at one point, contact at two points.]

Ex. 5. The conics $S=0$, $S'=0$ have a common self-conjugate triangle if and only if the invariant-factors of the determinant of $S-\lambda S'$ are all linear; and similarly for conicoids.

§ 10. Small Oscillations about a Position of Equilibrium.

Suppose a dynamical system is oscillating about a position of equilibrium.

Let the configuration of the system be known when the values of the 'generalized coordinates' $x_1, x_2, \ldots, x_m$ are given; which coordinates vanish when the system is in its position of equilibrium.

Then the kinetic energy T of the system is approximately of the form $\Sigma\, b_{ij}\dot{x}_i\dot{x}_j$, and the work U done by the forces acting on the system when it is displaced from the position of equilibrium to its actual position is approximately of the form $\Sigma\, a_{ij}x_ix_j$.*

Now $T \equiv \Sigma\, b_{ij}\dot{x}_i\dot{x}_j$ is essentially positive.

Hence (Ch. III, § 6) we can find real linear functions $\xi_1, \xi_2, \ldots, \xi_m$ of $x_1, x_2, \ldots, x_m$ such that T becomes

$$\dot{\xi}_1^2+\dot{\xi}_2^2+\ldots+\dot{\xi}_m^2,$$

while U becomes a quadratic function of $\xi_1, \xi_2, \ldots, \xi_m$.

Now (Ch. III, § 2) we can find real linear functions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ of $\xi_1, \xi_2, \ldots, \xi_m$, and therefore of $x_1, x_2, \ldots, x_m$ such that $T \equiv \dot{\xi}_1^2+\dot{\xi}_2^2+\ldots+\dot{\xi}_m^2$ becomes $\dot{\mathbf{x}}_1^2+\dot{\mathbf{x}}_2^2+\ldots+\dot{\mathbf{x}}_m^2$, while U becomes $\lambda_1\mathbf{x}_1^2+\lambda_2\mathbf{x}_2^2+\ldots+\lambda_m\mathbf{x}_m^2$.

By Ch. III, § 1, the determinants of the forms

$$\Sigma\,\lambda_i\mathbf{x}_i^2-\lambda\Sigma\mathbf{x}_i^2 \text{ and } \Sigma\, a_{ij}x_ix_j-\lambda\Sigma\, b_{ij}x_ix_j$$

are the same except for a constant multiplier.

* See Routh's *Rigid Dynamics*, i, Ch. IX. $a_{ij}=a_{ji}$, $b_{ij}=b_{ji}$. Dots denote differentiation with respect to the *time* t.

Hence $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the roots of the equation

$$\begin{vmatrix} a_{11}-\lambda b_{11} & a_{12}-\lambda b_{12} & \cdot & \cdot & \cdot & a_{1m}-\lambda b_{1m} \\ a_{21}-\lambda b_{21} & a_{22}-\lambda b_{22} & \cdot & & \cdot & a_{2m}-\lambda b_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1}-\lambda b_{m1} & a_{m2}-\lambda b_{m2} & \cdot & \cdot & \cdot & a_{mm}-\lambda b_{mm} \end{vmatrix} = 0.$$

Now applying Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{x}}_i}\right) - \frac{\partial T}{\partial \mathbf{x}_i} = \frac{\partial U}{\partial \mathbf{x}_i}$$

to $T \equiv \dot{\mathbf{x}}_1^2 + \dot{\mathbf{x}}_2^2 + \ldots + \dot{\mathbf{x}}_m^2$, $U \equiv \lambda_1 \mathbf{x}_1^2 + \lambda_2 \mathbf{x}_2^2 + \ldots + \lambda_m \mathbf{x}_m^2$,

we get $\ddot{\mathbf{x}}_i = \lambda_i \mathbf{x}_i \ (i = 1, 2, \ldots, m)$;

whence $\mathbf{x}_i = A_i \cos(\sqrt{-\lambda_i}\, t + \alpha_i)$, if λ_i is negative,

or $\mathbf{x}_i = A_i \cosh(\sqrt{\lambda_i}\, t + \alpha_i)$, if λ_i is positive,

where A_i and α_i are constants.

Hence the system, when slightly displaced in any way from the position of equilibrium, will oscillate about that position with a motion compounded of simple harmonic motions of periods

$$2\pi/\sqrt{-\lambda_1},\ 2\pi/\sqrt{-\lambda_2},\ \ldots,\ 2\pi/\sqrt{-\lambda_m},$$

if and only if $\lambda_1, \lambda_2, \ldots, \lambda_m$ are all negative; i.e. if U is a maximum in the position of equilibrium (§ 6).

If one or more of the quantities $\lambda_1, \lambda_2, \ldots, \lambda_m$ is positive, the position of equilibrium is unstable.

Quantities such as $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ satisfying a relation of the form $\ddot{\mathbf{x}}_i = \lambda_i \mathbf{x}_i$ are called 'principal coordinates' of the motion near the position of equilibrium. They may be more easily obtained in practice by applying Lagrange's equations

to $T \equiv \Sigma\, b_{ij} \dot{x}_i \dot{x}_j$ and $U \equiv \Sigma\, a_{ij} x_i x_j$.

We thus get m equations

$$b_{i1}\ddot{x}_1 + b_{i2}\ddot{x}_2 + \ldots + b_{im}\ddot{x}_{im} = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{im}x_{im},$$

which may be solved as in § 1.

Ex. A uniform rod of length $2a$ hangs from a fixed point by an inelastic thread of length $\frac{4}{5}a$ fastened to one end of the rod. Find the periods of small oscillation in a vertical plane about the vertical position of equilibrium.

[Let m be the mass of the rod, θ and ϕ the angles which the thread and rod make with the vertical.

Then
$$T \equiv ma^2 \left[\tfrac{8}{25}\dot\theta^2 + \tfrac{4}{5}\dot\theta\dot\phi\cos(\theta-\phi) + \tfrac{2}{3}\dot\phi^2\right],$$
$$U \equiv mag\left(\tfrac{4}{5}\cos\theta + \cos\phi - \tfrac{9}{5}\right);$$
or approximately, since θ and ϕ are small,
$$T \equiv ma^2\left(\tfrac{8}{25}\dot\theta^2 + \tfrac{4}{5}\dot\theta\dot\phi + \tfrac{2}{3}\dot\phi^2\right),\quad U \equiv mag\left(-\tfrac{2}{5}\theta^2 - \tfrac{1}{2}\phi^2\right).$$
Hence the periods are $2\pi/\sqrt{-\lambda_1}$ and $2\pi/\sqrt{-\lambda_2}$, where λ_1 and λ_2 are the roots of
$$\begin{vmatrix} -\tfrac{2}{5}g - \tfrac{8}{25}a\lambda & -\tfrac{2}{5}a\lambda \\ -\tfrac{2}{5}a\lambda & -\tfrac{1}{2}g - \tfrac{2}{3}a\lambda \end{vmatrix} = 0, \text{ i.e. } (2a\lambda + 15g)(2a\lambda + g) = 0.]$$

Other examples will be found in text-books on Mechanics.

§ 11. Thomson's and Bertrand's Theorems.

If, as in § 10,
$$T \equiv \Sigma\, b_{ij}\dot x_i \dot x_j \quad (i, j = 1, 2, \ldots, m)$$
is the kinetic energy of a moving system,
$$\begin{vmatrix} b_{11} & b_{12} & . & . & . & b_{1k} \\ b_{21} & b_{22} & . & . & . & b_{2k} \\ . & . & . & . & . & . \\ b_{k1} & b_{k2} & . & . & . & b_{kk} \end{vmatrix} \neq 0.$$

For otherwise T would vanish when
$$\dot x_{k+1} = \dot x_{k+2} = \ldots = \dot x_m = 0,$$
but not all of $\dot x_1, \dot x_2, \ldots, \dot x_k = 0$; which is impossible from the dynamical meaning of T.

We can therefore, by Ch. III, § 4, express T as the sum $T_1 + T_2$ of a quadratic function T_1 in $\dot\xi_1, \dot\xi_2, \ldots, \dot\xi_k$ and a quadratic function T_2 in $\dot x_{k+1}, \dot x_{k+2}, \ldots, \dot x_m$, where
$$\xi_t = b_{t1}x_1 + b_{t2}x_2 + \ldots + b_{tm}x_m \quad (t = 1, 2, \ldots, k).$$
Neither T_1 nor T_2 can be ≤ 0 for non-zero values of the variables concerned by the dynamical meaning of T.

Suppose now the system was started from rest by impulses. We may choose our generalized coordinates so that the velocities of the points at which the impulses are applied are given by the values of $\dot x_{k+1}, \dot x_{k+2}, \ldots, \dot x_m$ while the values of $\dot x_1, \dot x_2, \ldots, \dot x_k$ or of $\dot\xi_1, \dot\xi_2, \ldots, \dot\xi_k$ give the initial velocities of other points of the system.

Applying Lagrange's impulse-equations to T when T is

expressed in the form T_1+T_2, we get in the actual initial motion

$$\dot{\xi}_1 = \dot{\xi}_2 = \ldots = \dot{\xi}_k = 0, \text{ so that } T_1 = 0.$$

Suppose the system had been started in any other manner so that the points of application had the same initial velocities. Then T_2 would have the same value as before, but T_1 is ≥ 0.

Hence we have *Thomson's Theorem* :—

'If a system is started from rest by impulses, the initial kinetic energy is less in the actual motion than in any possible motion in which the points of application of the impulses have the same velocities.' *

Similarly we have *Bertrand's Theorem* :—

'If a system is acted on by impulses, then the kinetic energy of the system in the actual subsequent motion is initially greater than if the system had been subjected to additional smooth constraints and acted on by the same impulses.'

We may suppose that the generalized coordinates were so chosen that $x_{k+1}, x_{k+2}, \ldots, x_m$, which can vary in the actual motion, must be constants when the additional smooth constraints are added. Putting T in the form T_1+T_2 as before, we see that Lagrange's impulse-equations give the same values of $\xi_1, \xi_2, \ldots, \xi_k$ and therefore of T_1 whether the additional smooth constraints are added or not. But T_2 is zero when the constraints act, and is greater than zero (in general) in the actual motion.

Ex. A uniform rod AB of length $2a$ and mass m rests on a smooth horizontal table. If it is struck by a blow of magnitude P at A perpendicular to its length, about what point will the rod begin to turn?

[(i) Suppose A begins to move with velocity v and the rod to turn with angular velocity ω. Its kinetic energy is

$$m\left\{\frac{v^2}{8}+\frac{1}{24}(4a\omega-3v)^2\right\},$$

which is a minimum, taking v fixed, when $\omega = 3v/4a$. This is therefore the relation between ω and v in the actual initial motion; so that the rod begins to turn about a point dividing AB in the ratio $2:1$.

* It is sufficient to prescribe the component-velocity of each point of application in the direction of the corresponding impulse.

(ii) Suppose the rod pivoted at a point C distant $a+x$ from A. Taking moments about C we have $P(a+x)=\frac{m}{3}(a^2+3x^2)\omega$, so that the kinetic energy of the rod is

$$\frac{m}{6}(a^2+3x^2)\,\omega^2=\frac{3P^2(a+x)^2}{2m(a^2+3x^2)}.$$

This is a maximum, taking P fixed, when $x=\frac{1}{3}a$; so that the rod begins to turn about a point distant $a+\frac{1}{3}a$ from A, as found in (i).]

Other examples will be found in text-books on Mechanics.

CHAPTER V

SUBSTITUTIONS PERMUTABLE WITH A GIVEN SUBSTITUTION

§ 1.

GIVEN a substitution A, we shall find all substitutions permutable with A. They are infinite in number.

If B is any substitution permutable with A, so that $AB = BA$, then $S^{-1}AS \, . \, S^{-1}BS = S^{-1}BS \, . \, S^{-1}AS$, or $S^{-1}BS$ is permutable with $S^{-1}AS$. It suffices therefore to find the substitutions permutable with any substitution N into which A may be transformed. The substitutions permutable with N will be transformed into the substitutions permutable with A by the substitution transforming N into A.

We shall take for N the canonical substitution into which any substitution may be transformed (Ch. I, § 9).

It is $\qquad x_i' = \lambda_i x_i + \beta_i x_{i+1} \; (i = 1, 2, \ldots, m),$

where $\beta_i = 1$ or 0, and is certainly 0 if $\lambda_i \neq \lambda_{i+1}$ $(\beta_m = 0)$.

§ 2. Substitutions permutable with a Canonical Substitution.

Let C

$$x_t' = c_{t1}x_1 + c_{t2}x_2 + \ldots + c_{tm}x_m \; (t = 1, 2, \ldots, m)$$

be any substitution permutable with N.

Equating the elements in the i-th row and j-th column of the matrices of CN and NC, we have

$$c_{ij}\lambda_j + c_{ij-1}\beta_{j-1} = c_{ij}\lambda_i + c_{i+1j}\beta_i, \ldots\ldots\ldots\ldots\text{(i)}$$

or writing out in full for $i = 1, 2, \ldots, m$

$$\left.\begin{aligned}
c_{1j}(\lambda_1 - \lambda_j) &= c_{1j-1}\beta_{j-1} - c_{2j}\beta_1 \\
c_{2j}(\lambda_2 - \lambda_j) &= c_{2j-1}\beta_{j-1} - c_{3j}\beta_2 \\
&\vdots \\
c_{m-1j}(\lambda_{m-1} - \lambda_j) &= c_{m-1j-1}\beta_{j-1} - c_{mj}\beta_{m-1} \\
c_{mj}(\lambda_m - \lambda_j) &= c_{mj-1}\beta_{j-1}
\end{aligned}\right\} \ldots\ldots\ldots\text{(ii)}$$

Suppose now, for example,

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, $\beta_1 = \beta_2 = \beta_3 = 1$, $\beta_4 = 0$; but $\lambda_1 \neq \lambda_5, \lambda_6, \ldots$.

Taking $j = 5, 6, 7, \ldots$ the equations (ii) give

$$\left.\begin{aligned} c_{15}(\lambda_1-\lambda_5) &= -c_{25} \\ c_{25}(\lambda_1-\lambda_5) &= -c_{35} \\ c_{35}(\lambda_1-\lambda_5) &= -c_{45} \\ c_{45}(\lambda_1-\lambda_5) &= 0 \end{aligned}\right\}, \quad \left.\begin{aligned} c_{16}(\lambda_1-\lambda_6) &= c_{15}\beta_5 - c_{26} \\ c_{26}(\lambda_1-\lambda_6) &= c_{25}\beta_5 - c_{36} \\ c_{36}(\lambda_1-\lambda_6) &= c_{35}\beta_5 - c_{46} \\ c_{46}(\lambda_1-\lambda_6) &= c_{45}\beta_5 \end{aligned}\right\}, \ldots$$

whence

$$c_{15} = c_{25} = c_{35} = c_{45} = 0, \quad c_{16} = c_{26} = c_{36} = c_{46} = 0, \ldots$$

The method is general and gives us $c_{ij} = 0$, whenever $\lambda_i \neq \lambda_j$.

Hence if N is expressed as the direct product of constituents $N_1, N_2, N_3, \ldots$ whose characteristic-determinants are $(\alpha-\lambda)^a, (\beta-\lambda)^b, (\gamma-\lambda)^c, \ldots$, where no two of $\alpha, \beta, \gamma, \ldots$ are equal, then any substitution permutable with N is the direct product of a substitution on the variables affected by N_1, a substitution on the variables affected by N_2, a substitution on the variables affected by $N_3, \ldots$.

Of course, N_1 is in general itself the direct product of substitution of the type

$$x_1' = \alpha x_1 + x_2, \ldots, \; x'_{r-1} = \alpha x_{r-1} + x_r, \; x_r' = \alpha x_r;$$

but each of the constituents of N_1 has only α as characteristic-root; and so for $N_2, N_3, \ldots$.

If $\lambda_i = \lambda_j$, we have from (i)

$$c_{i\,j-1}\beta_{j-1} = c_{i+1\,j}\beta_i.$$

Hence

$$\left.\begin{aligned} c_{i\,j-1} &= c_{i+1\,j} && \text{when } \beta_i = 1, \; \beta_{j-1} = 1 \\ c_{i\,j-1} &= 0 && \text{when } \beta_i = 0, \; \beta_{j-1} = 1 \\ c_{i+1\,j} &= 0 && \text{when } \beta_i = 1, \; \beta_{j-1} = 0 \end{aligned}\right\}.^* \quad \ldots\ldots\ldots \text{(iii)}$$

Consider, for instance, the case $m = 8$,

$$\lambda_1 = \lambda_2 = \ldots = \lambda_8 = \alpha, \; \beta_1 = \beta_2 = \beta_3 = \beta_4 = 1, \; \beta_5 = 0,$$
$$\beta_6 = \beta_7 = 1, \; \beta_8 = 0,$$

so that N has the matrix †

$$\left|\begin{matrix} \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{matrix}\right| \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}$$

* In these equations $\beta_0 = \beta_m = 0$.

† The zeros are put in small type to emphasize the non-zero elements.

$\beta_i \;\; = 1$ when $i = 1, 2, 3, 4, 6, 7$; $\beta_i \;\; = 0$ when $i = 5, 8$.
$\beta_{j-1} = 1$ when $j = 2, 3, 4, 5, 7, 8$; $\beta_{j-1} = 0$ when $j = 1, 6$.
Hence from (iii)

$c_{11} = c_{22} = c_{33} = c_{44} = c_{55}$, $c_{12} = c_{23} = c_{34} = c_{45}$, $c_{13} = c_{24} = c_{35}$, $c_{14} = c_{25}$.

$0 = c_{21} = c_{32} = c_{43} = c_{54}$, $0 = c_{31} = c_{42} = c_{53}$, $0 = c_{41} = c_{52}$, $0 = c_{51}$.

$c_{66} = c_{77} = c_{88}$, $c_{67} = c_{78}$.

$0 = c_{76} = c_{87}$, $0 = c_{86}$.

$c_{16} = c_{27} = c_{38}$, $c_{17} = c_{28}$.

$0 = c_{26} = c_{37} = c_{48}$, $0 = c_{36} = c_{47} = c_{58}$. $0 = c_{46} = c_{57}$, $0 = c_{56}$.

$c_{63} = c_{74} = c_{85}$, $c_{64} = c_{75}$.

$0 = c_{62} = c_{73} = c_{84}$, $0 = c_{61} = c_{72} = c_{83}$, $0 = c_{71} = c_{82}$, $0 = c_{81}$.

Hence the matrix of any substitution permutable with N is of the type

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ 0 & a & b & c & d & 0 & f & g \\ 0 & 0 & a & b & c & 0 & 0 & f \\ 0 & 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & l & m & n & p & q & r \\ 0 & 0 & 0 & l & m & 0 & p & q \\ 0 & 0 & 0 & 0 & l & 0 & 0 & p \end{vmatrix} \quad \text{..................(v)}$$

The reader will notice the diagonal arrangement of equal elements in the matrix, and the positions of the zeros.

As another illustration, take N with the matrix

$$\begin{vmatrix} \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{vmatrix} \quad \text{..................(vi)}$$

The general substitution permutable with N has the matrix

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ 0 & a & b & 0 & d & e & 0 & g \\ 0 & 0 & a & 0 & 0 & d & 0 & 0 \\ i & j & k & l & m & n & p & q \\ 0 & i & j & 0 & l & m & 0 & p \\ 0 & 0 & i & 0 & 0 & l & 0 & 0 \\ 0 & r & s & 0 & t & u & v & w \\ 0 & 0 & r & 0 & 0 & t & 0 & v \end{vmatrix} \quad \text{..................(vii)}$$

We see that, if N has a pair of constituent substitutions of the types

$$\left.\begin{array}{l} x_1' = \alpha x_1 + x_2,\ \ldots,\ x'_{r-1} = \alpha x_{r-1} + x_r,\ x_r' = \alpha x_r \\ x_1' = \alpha x_1 + x_2,\ \ldots,\ x'_{s-1} = \alpha x_{s-1} + x_s,\ x_s' = \alpha x_s \end{array}\right\},\ r \geq s,$$

then the matrix of the general substitution permutable with N has corresponding square arrays of the form

$$\begin{bmatrix} a_1 & a_2 & a_3 & . & . & . & a_s \\ 0 & a_1 & a_2 & . & . & . & a_{s-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & a_1 \\ 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & . & . & . & 0 & b_1 & b_2 & b_3 & . & . & . & b_s \\ 0 & . & . & . & 0 & 0 & b_1 & b_2 & . & . & . & b_{s-1} \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & 0 & 0 & 0 & 0 & . & . & . & b_1 \end{bmatrix},$$

each with s arbitrary coefficients.

It very readily follows that the substitution A of Ch. II, § 5, Corollary I, with invariant-factors

$$\begin{array}{ll} (\lambda-\alpha)^{a_1},\ (\lambda-\alpha)^{a_2},\ (\lambda-\alpha)^{a_3},\ \ldots, & \text{where } a_1 \geq a_2 \geq a_3 \geq \ldots, \\ (\lambda-\beta)^{b_1},\ (\lambda-\beta)^{b_2},\ (\lambda-\beta)^{b_3},\ \ldots, & \text{where } b_1 \geq b_2 \geq b_3 \geq \ldots, \\ (\lambda-\gamma)^{c_1},\ (\lambda-\gamma)^{c_2},\ (\lambda-\gamma)^{c_3},\ \ldots, & \text{where } c_1 \geq c_2 \geq c_3 \geq \ldots, \\ . \quad . \quad . \quad . \quad . \quad . \quad . & . \quad . \quad . \quad . \quad . \quad . \quad . \end{array}$$

is permutable with a k-ply infinite number of substitutions where

$$\begin{aligned} k = {} & (a_1+a_2+a_3+a_4+\ldots)+2\,(a_2+2a_3+3a_4+\ldots) \\ & +(b_1+b_2+b_3+b_4+\ldots)+2\,(b_2+2b_3+3b_4+\ldots) \\ & + \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

or

$$k = (a_1+3a_2+5a_3+7a_4+\ldots)+(b_1+3b_2+5b_3+7b_4+\ldots)+\ldots.$$

Ex. 1. Any substitution permutable with

$$x_1' = \alpha x_1 + x_2,\ x_2' = \alpha x_2 + x_3,\ \ldots,\ x'_{m-1} = \alpha x_{m-1} + x_m,\ x_m' = \alpha x_m$$

is of the type given in Ch. I, § 2, Ex. 8; and any two such substitutions are themselves permutable.

Ex. 2. The substitutions permutable with

$$x_1' = x_2,\ x_2' = x_3,\ \ldots,\ x'_{m-1} = x_m,\ x_m' = e_1 x_1 + e_2 x_2 + \ldots + e_m x_m$$

are m-ply infinite in number.

Ex. 3. If *every* substitution permutable with A is of the form $p_0 A^0 + p_1 A^1 + p_2 A^2 + p_3 A^3 + \ldots$, a *single* invariant-factor of A corresponds to each distinct characteristic-root of A; and conversely.*

[It is sufficient to prove the theorem when A is in canonical form.]

Ex. 4. The only substitutions on $x_1, x_2, \ldots, x_m$ permutable with every permutation on $x_1, x_2, \ldots, x_m$ are those of the form

$$x_t' = \beta x_1 + \ldots + \beta x_{t-1} + \alpha x_t + \beta x_{t+1} + \ldots + \beta x_m \quad (t = 1, 2, \ldots, m).$$

[If A is permutable with the 'transposition'

$$x_1' = x_2,\ x_2' = x_1,\ x_3' = x_3,\ \ldots,\ x_m' = x_m,$$

$a_{11} = a_{22},\ a_{12} = a_{21},\ a_{13} = a_{23},\ a_{14} = a_{24},\ \ldots,\ a_{31} = a_{32},\ a_{41} = a_{42},\ \ldots$. Hence, since A is permutable with *every* transposition, the result follows.]

Ex. 5. If A has a *single* invariant-factor, there are exactly q substitutions B such that $B^q = A$.

[It is sufficient to take A in canonical form C

$$x_1' = \alpha x_1 + x_2,\ \ldots,\ x'_{m-1} = \alpha x_{m-1} + x_m,\ x_m' = \alpha x_m.$$

Now if $D^q = C$, D is permutable with C and therefore takes the form

$$x_1' = a x_1 + b x_2 + c x_3 + d x_4 + \ldots,\ x_2' = a x_2 + b x_3 + c x_4 + \ldots,$$
$$\ldots,\ x_m' = a x_m.$$

The q-th power of this substitution is given in Ch. I, § 3, Ex. 13. Identifying it with C, we have

$$a^q = \alpha,\ q a^{q-1} b = 1,\ q c a^{q-1} + \tfrac{1}{2} q (q-1) b^2 a^{q-2} = 0,\ \ldots,$$

which gives q values for a, and, when a is chosen, unique values for $b, c, d, \ldots$.

It follows from Ch. I, § 3, Ex. 13, that $D^q = C$, if D is any one of the q substitutions

$$x_1' = \alpha^k x_1 + k_1 \alpha^{k-1} x_2 + k_2 \alpha^{k-2} x_3 + \ldots,$$
$$x_2' = \alpha^k x_2 + k_1 \alpha^{k-1} x_2 + k_2 \alpha^{k-2} x_3 + \ldots,\ \ldots,\ x_m' = \alpha^k x_m,$$

where $k_r \equiv k(k-1)\ldots(k-r+1) \div r!$, and $kq = 1$.]

* This result is due to Cecioni: *Atti Reale Accad. dei Lincei* xviii (1909), p. 566.

Ex. 6. Discuss the case in which A has more than one distinct characteristic-root, but only a single invariant-factor corresponding to each characteristic-root.

Ex. 7. If A has more than one invariant-factor corresponding to a given characteristic-root, the number of substitutions B such that $B^q = A$ is infinite.

Ex. 8. Find the value of k if k-ply infinite substitutions are permutable with A, where A is the substitution of Ch. I, § 6, Ex. 6, 7; § 9, Ex. 1, 3, 4, 5, 6.

Ex. 9. Find the substitutions on $x_1, x_2, \ldots, x_m$ permutable with $(x_m, x_{m-1}, \ldots, x_2, x_1)$.

[The matrix of such a substitution is symmetrical about its centre. Hence their number is $\frac{1}{2}m^2$-ply infinite when m is even, and $\frac{1}{2}(m^2+1)$-ply infinite when m is odd.
Verify that this agrees with the result at the end of § 2.]

Ex. 10. Find the substitutions on $x_1, x_2, \ldots, x_m$ permutable with $(x_2, x_3, \ldots, x_m, x_1)$.

[Any cyclant substitution of type I; see Ch. I, § 2, Ex. 7. Hence their number is m-ply infinite.
Verify that this agrees with the result at the end of § 2.]

Ex. 11. Find the substitutions permutable with $(ax - by, bx + ay)$, where $b \neq 0$.

[Any substitution of the same type.]

Ex. 12. Show that the determinant (vii) of § 2 factorizes into

$$\pm \begin{vmatrix} a & d \\ i & l \end{vmatrix}^3 \times v^2,$$

and that a similar process applies in general.

[Cf. Ch. VI, § 4, Corollary II.]

Ex. 13. If the substitution A of § 2 can be transformed into a given substitution B, it can be so transformed by a k-ply infinite number of substitutions.

§ 3. Substitutions permutable with every Substitution permutable with a given Substitution.

We shall now find the substitutions permutable with every substitution permutable with the given substitution A at the end of the last section, and show that they are k-ply infinite in number, where

$$k = a_1 + b_1 + c_1 + \ldots.$$

As in § 1 it will suffice to take A in canonical form N.
Suppose, for example, that N has the matrix (vi) of § 2.

One substitution permutable with N has the matrix

$$\begin{vmatrix} \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma \end{vmatrix} \quad \text{..................(viii)}$$

and therefore all the required substitutions, being permutable with the substitution with matrix (viii), have a matrix of the type obtained from (vii) by putting $d, e, f, g, h, i, j, k, p, q, r, s, t, u$ all zero.

Moreover, since the substitution now obtained has to be permutable with the substitution with the matrix obtained from (vii) by putting d, e, f, g, h, p, q all unity, and i, j, k, r, s, t, u all zero, we readily prove that in the required matrix $a = l = v$, $b = m = w$, $c = n$; so that a substitution permutable with every substitution with matrix of the type (vii) is of the type

$$\begin{vmatrix} a & b & c & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & c & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{vmatrix} \quad \text{.................. (ix)}$$

The reader will notice the diagonal arrangement of the equal elements.

Conversely, it is at once verified that every substitution with matrix of the type (ix) is permutable with every substitution of the type (vii).

The method used is general and establishes the result stated at the beginning of the section.

As another example, every substitution permutable with all substitutions with matrix of the type (v) has a matrix of the type

$$\begin{vmatrix} a & b & c & d & e & 0 & 0 & 0 \\ 0 & a & b & c & d & 0 & 0 & 0 \\ 0 & 0 & a & b & c & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{vmatrix}.$$

CHAPTER VI

SYMMETRIC, ALTERNATE, AND HERMITIAN SUBSTITUTIONS.

§ 1. Expression of a Symmetric Substitution in the form BB′.

We showed in Ch. I, § 10, that, if A is a symmetric substitution and B is any substitution, $B'AB$ is symmetric.

In particular, taking A as the unit substitution E, we have $B'B$ (and BB') symmetric.

Conversely, given any symmetric substitution A we can find B so that $A = BB'$.

For suppose that $\Sigma\, a_{ij} x_i x_j$ when expressed as the sum of m squares, as in Ch. III, § 6, takes the form

$$(b_{11}x_1 + b_{12}x_2 + \dots + b_{1m}x_m)^2 + (b_{21}x_1 + b_{22}x_2 + \dots + b_{2m}x_m)^2 + \dots \\ + (b_{m1}x_1 + b_{m2}x_2 + \dots + b_{mm}x_m)^2.$$

Then B is the substitution

$$x_t' = b_{t1}x_1 + b_{t2}x_2 + \dots + b_{tm}x_m \quad (t = 1, 2, \dots, m).$$

For if we operate with this substitution on

$$x_1^2 + x_2^2 + \dots + x_m^2,$$

we get $\Sigma\, a_{ij} x_i x_j$.

But the result of operating in this way is, by Ch. III, § 1, the quadratic form corresponding to the substitution BEB' or BB'.

Similarly, if A is a *positive* Hermitian substitution, we can find B so that $A = B\bar{B}'$.

If $\Sigma\, a_{ij} \bar{x}_i x_j$ is not positive, but can be transformed into the type (Ch. III, § 2)

$$\xi_1\bar{\xi}_1 + \dots + \xi_k\bar{\xi}_k - \xi_{k+1}\bar{\xi}_{k+1} - \dots - \xi_m\bar{\xi}_m,$$

we can find B so that $A = BM\bar{B}'$, where M is the multiplication

$$x_1' = x_1, \dots, x_k' = x_k, x'_{k+1} = -x_{k+1}, \dots, x_m' = -x_m.$$

Ex. 1. Find B so that $BB' = A$, where A is

$$(x-y+z,\ -x-2z,\ x-2y+z).$$

[$x^2+z^2-4yz+2zx-2xy \equiv (x-y+z)^2+(iy+iz)^2+z^2$.

$$\therefore\quad A = BB', \text{ where } B \equiv (x-y+z,\ iy+iz,\ z),$$

as is immediately verified.]

Ex. 2. Find B so that $BB' = A$, where A is

$$(x-y,\ -x+5y),\ (x+y+5z,\ x+2z,\ 5x+2y),$$
$$(x+y+w,\ x-z,\ -y-2w,\ x-2z+2w).$$

Ex. 3. Show that if A and C are two given symmetric substitutions, we can find a substitution B so that $A = BCB'$.

[If $A = PP'$ and $C = QQ'$, $B = PQ^{-1}$.]

Ex. 4. If A is $(x+y,\ x+10y)$ and C is $(4x-2y,\ -2x+5y)$, find B so that $A = BCB'$.

Ex. 5. The product of two Hermitian substitutions, one of which is positive, is transformable into a multiplication.

[If A and B are Hermitian, and B is positive, we can choose Q so that $QB\bar{Q}' = E$.

Then AB is transformable into $\bar{Q}'^{-1}AB\bar{Q}' = \bar{Q}'^{-1}AQ^{-1}$, which is Hermitian and therefore transformable into a multiplication.]

Ex. 6. The product of two real symmetric substitutions, one of which is positive, is transformable into a multiplication.

Ex. 7. Find B so that $A = BM\bar{B}'$, where

$$A \equiv (x_4,\ x_3,\ x_2,\ x_1) \text{ and } M \equiv (x_1,\ x_2,\ -x_3,\ -x_4).$$

[$x_1\bar{x}_4 + x_2\bar{x}_3 + x_3\bar{x}_2 + x_4\bar{x}_1 \equiv (x_1+\tfrac{1}{2}x_4)(\bar{x}_1+\tfrac{1}{2}\bar{x}_4)$
$+ (x_2+\tfrac{1}{2}x_3)(\bar{x}_2+\tfrac{1}{2}\bar{x}_3) - (x_2-\tfrac{1}{2}x_3)(\bar{x}_2-\tfrac{1}{2}\bar{x}_3) - (x_1-\tfrac{1}{2}x_4)(\bar{x}_1-\tfrac{1}{2}\bar{x}_4)$.

Hence $\quad B \equiv (x_1+\tfrac{1}{2}x_4,\ x_2+\tfrac{1}{2}x_3,\ x_2-\tfrac{1}{2}x_3,\ x_1-\tfrac{1}{2}x_4)$.]

§ 2. A Condition that N should be Transformable into A, where AK is Symmetric.

Suppose $P^{-1}NP = A$ and $PKP' = C$. Then, if AK is symmetric, so is NC; and, conversely, if NC is symmetric, so is AK.

We adopt the usual notation, so that A is the substitution

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \dots + a_{tm}x_m \quad (t = 1, 2, \dots, m),$$

and so for P, N, C, K.

Then by Ch. I, § 2, $c_{ij} = \sum_{\sigma,\tau} p_{\sigma i}p_{\tau j}k_{\sigma\tau}$.

Suppose AK symmetric, so that for all values of σ and t

$$k_{\sigma 1}a_{1t} + k_{\sigma 2}a_{2t} + \dots + k_{\sigma m}a_{mt} = k_{t1}a_{1\sigma} + k_{t2}a_{2\sigma} + \dots + k_{tm}a_{m\sigma}.$$

Also, since $NP = PA$, we have for all values of i and t

$$p_{t1}n_{1i}+p_{t2}n_{2i}+\ldots+p_{tm}n_{mi} = a_{t1}p_{1i}+a_{t2}p_{2i}+\ldots+a_{tm}p_{mi}.$$

Then

$$\begin{aligned}
&c_{i1}n_{1j}+c_{i2}n_{2j}+\ldots+c_{im}n_{mj}\\
&\quad= \sum_{\sigma,\tau} p_{\sigma i}k_{\sigma\tau}(p_{\tau 1}n_{1j}+p_{\tau 2}n_{2j}+\ldots+p_{\tau m}n_{mj})\\
&\quad= \sum_{\sigma,\tau} p_{\sigma i}k_{\sigma\tau}(a_{\tau 1}p_{1j}+a_{\tau 2}p_{2j}+\ldots+a_{\tau m}p_{mj})\\
&\quad= \sum_{\sigma} p_{\sigma i}\{p_{1j}(k_{\sigma 1}a_{11}+k_{\sigma 2}a_{21}+\ldots+k_{\sigma m}a_{m1})+\ldots\\
&\qquad\qquad\qquad +p_{mj}(k_{\sigma 1}a_{1m}+k_{\sigma 2}a_{2m}+\ldots+k_{\sigma m}a_{mm})\}\\
&\quad= \sum_{\sigma} p_{\sigma i}\{p_{1j}(k_{11}a_{1\sigma}+k_{12}a_{2\sigma}+\ldots+k_{1m}a_{m\sigma})+\ldots\\
&\qquad\qquad\qquad +p_{mj}(k_{m1}a_{1\sigma}+k_{m2}a_{2\sigma}+\ldots+k_{mm}a_{m\sigma})\}\\
&\quad= \sum_{\sigma} p_{\sigma i}\{a_{1\sigma}(p_{1j}k_{11}+p_{2j}k_{21}+\ldots+p_{mj}k_{m1})+\ldots\\
&\qquad\qquad\qquad +a_{m\sigma}(p_{1j}k_{1m}+p_{2j}k_{2m}+\ldots+p_{mj}k_{mm})\}\\
&\quad= \sum_{\sigma} n_{\sigma i}\{p_{1\sigma}(p_{1j}k_{11}+p_{2j}k_{21}+\ldots+p_{mj}k_{m1})+\ldots\\
&\qquad\qquad\qquad +p_{m\sigma}(p_{1j}k_{1m}+p_{2j}k_{2m}+\ldots+p_{mj}k_{mm})\}\\
&\quad= c_{j1}n_{1i}+c_{j2}n_{2i}+\ldots+c_{jm}n_{mi},
\end{aligned}$$

so that NC is symmetric.

Conversely, if NC is symmetric, we have for all values of i and j

$$c_{i1}n_{1j}+c_{i2}n_{2j}+\ldots+c_{im}n_{mj} = c_{j1}n_{1i}+c_{j2}n_{2i}+\ldots+c_{jm}n_{mi},$$

whence, by the above reasoning, for all values of σ and t

$$\sum_{\sigma,t} p_{\sigma i}p_{tj}\{(k_{\sigma 1}a_{1t}+k_{\sigma 2}a_{2t}+\ldots+k_{\sigma m}a_{mt}) \\ -(k_{t1}a_{1\sigma}+k_{t2}a_{2\sigma}+\ldots+k_{tm}a_{m\sigma})\} = 0.$$

The reader will readily prove that the determinant of these m^2 linear equations in the m^2 quantities

$$(k_{\sigma 1}a_{1t}+k_{\sigma 2}a_{2t}+\ldots+k_{\sigma m}a_{mt})-(k_{t1}a_{1\sigma}+k_{t2}a_{2\sigma}+\ldots+k_{tm}a_{m\sigma})$$

is the $2m$-th power of the determinant of P, and is therefore not zero. Hence each of the m^2 quantities is zero; or AK is symmetric.

§ 3. Applications of this Condition.

As a first illustration of § 1 suppose that K is the unit substitution E, and that P is orthogonal.

Then $C = PP' = E$; so that N is symmetric if A is. Hence the transform of a symmetric substitution by an orthogonal substitution is symmetric, as proved in Ch. I, § 10.

As another illustration of the case in which $K = E$, consider the problem:—

To find a symmetric substitution with given invariant-factors.

It will suffice to find a symmetric substitution with a single invariant-factor $(\lambda-\alpha)^r$. For if we find such a symmetric substitution for each of the given invariant-factors, their direct product will be the substitution required by Ch. II, § 4.

Now if N and C are respectively

$$x_1' = \alpha x_1 + x_2,\ \ldots,\ x'_{r-1} = \alpha x_{r-1} + x_r,\ x_r' = \alpha x_r,$$
$$x_1' = x_r,\quad \ldots,\ x'_{r-1} = x_2,\qquad x_r' = x_1,$$

then NC is

$$x_1' = \alpha x_r,\ x_2' = \alpha x_{r-1} + x_r,\ \ldots,\ x'_{r-1} = \alpha x_2 + x_3,\ x_r' = \alpha x_1 + x_2,$$

which is symmetric.

Find by § 1 a substitution P such that $C = PP'$. Then $P^{-1}NP$ is the required symmetric substitution with the single invariant-factor $(\lambda-\alpha)^r$.

Now, taking the general case in which K is not necessarily E, consider the problem:—

*To express a given substitution A as the product of two symmetric substitutions.**

Take N as the canonical substitution into which A may be transformed (Ch. I, § 9), namely the direct product of substitutions of the type

$$x_1' = \alpha x_1 + x_2,\ \ldots,\ x'_{r-1} = \alpha x_{r-1} + x_r,\ x_r' = \alpha x_r.$$

Let C be the direct product of substitutions such as

$$x_1' = x_r,\ \ldots,\ x'_{r-1} = x_2,\ x_r' = x_1.$$

As shown above, NC is symmetric; and therefore $AK = L$ is symmetric, where $PKP' = C$. But since C is symmetric, $K^{-1} = P'C^{-1}P$ is symmetric (Ch. I, § 10); and $A = L \,.\, K^{-1}$ is thus expressed as the product of two symmetric substitutions L and K^{-1}.

Consider now the problem:—

To find every possible manner of expressing a given substitution A as the product of two symmetric substitutions.

Suppose the two factors of the product are the symmetric substitutions L and K^{-1}, so that $AK = L$. We want to find every possible way of choosing K; for when A and K are given, so is L.

* That A can be so expressed was proved by Frobenius, *Berliner Sitzungsberichte*, 1910, p. 3.

It is sufficient to solve the problem for any substitution N into which A can be transformed. For let $A = P^{-1}NP$, where P and N are supposed given; and let C be any symmetric substitution such that NC is symmetric. Then $K = P^{-1}CP'^{-1}$ is a possible value of K, and each such value of C gives a single value of K.

We may take N in canonical form, and must now solve the problem :—

To find every symmetric substitution C such that NC is symmetric, N being a canonical substitution.

The solution is given in § 4.

Ex. 1. Find a symmetric substitution with invariant-factor $(\lambda-\alpha)^3$.

[$2x_1x_3+x_2^2 \equiv \left(\frac{x_1+x_3}{\sqrt{2}}\right)^2 + x_2^2+\left(\frac{ix_1-ix_3}{\sqrt{2}}\right)^2$. Therefore the substitution $C\equiv(x_3, x_2, x_1)$ is PP', where

$$P\equiv\left(\frac{x_1+x_3}{\sqrt{2}},\ x_2,\ \frac{ix_1-ix_3}{\sqrt{2}}\right).$$

And $N\equiv(\alpha x_1+x_2,\ \alpha x_2+x_3,\ \alpha x_3)$ is transformed by P into the required symmetric substitution

$$\left(\alpha x_1+\frac{x_2}{\sqrt{2}},\ \frac{x_1}{\sqrt{2}}+\alpha x_2+\frac{ix_3}{\sqrt{2}},\ \frac{ix_2}{\sqrt{2}}+\alpha x_3\right).]$$

Ex. 2. Find symmetric substitutions with invariant-factors $(\lambda-\alpha)^2$ and $(\lambda-\alpha)^4$.

[The matrices of the required substitutions are

$$\begin{vmatrix} \alpha+\frac{1}{2} & \frac{1}{2}i \\ \frac{1}{2}i & \alpha-\frac{1}{2} \end{vmatrix} \text{ and } \begin{vmatrix} \alpha & \frac{1}{2} & -\frac{1}{2}i & 0 \\ \frac{1}{2} & \alpha+\frac{1}{2} & \frac{1}{2}i & \frac{1}{2}i \\ -\frac{1}{2}i & \frac{1}{2}i & \alpha-\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2}i & \frac{1}{2} & \alpha \end{vmatrix}.$$

The reader will find no difficulty in extending the method to the case of an invariant-factor $(\lambda-\alpha)^r$ of any index r.]

Ex. 3. If $P^{-1}NP=A$, where A and N are symmetric, and $C=PP'$, then $NC=CN$.

[By § 2 $\qquad NC=(NC)'=C'N'=CN.$]

Ex. 4. Express $(3x+y-2z,\ 2x+3y-3z,\ 4x+2y-3z)$ as the product of two symmetric substitutions.

[$PAP^{-1}=N$ if $N\equiv(x+y,\ y+z,\ z)$

and $\qquad P\equiv(x,\ 2x+y+2z,\ 2x+z)$,

so that $\qquad P^{-1}\equiv(x,\ 2x+y-2z,\ -2x+z)$.

Hence, if $C \equiv (z, y, x)$,

$$K^{-1} \equiv P'C^{-1}P \text{ is } (2y+z,\ 2x+9y+6z,\ x+6y+4z)$$

and $$K \equiv (2y-3z,\ 2x+y-2z,\ -3x-2y+4z).$$

Then $AK \equiv L$ is $(-8x+3z,\ y-z,\ 3x-y)$ and A is the product of the two symmetric substitutions L and K^{-1}.]

Ex. 5. Express as the product of two symmetric substitutions the substitutions of Ch. I, § 9, Ex. 1, 3, 4, 5.

Ex. 6. Any substitution can be transformed into the transposed substitution by means of a symmetric substitution. If this can be done by a k-ply infinite number of symmetric substitutions, find k.

[If AK and K are symmetric, $K^{-1}AK = A'$.]

§ 4. Transformation of a Symmetric into a Canonical Substitution.

Take N of § 2 as the canonical substitution of Ch. V, § 1,

$$x_i' = \lambda_i x_i + \beta_i x_{i+1} \ (i = 1, 2, \ldots, m),$$

where $\beta_i = 1$ or 0, and is certainly 0 if

$$\lambda_i \neq \lambda_{i+1} \ (\beta_0 = \beta_m = 0).$$

Suppose NC and C are symmetric. Then

$$c_{ji}\lambda_i + c_{j\,i-1}\beta_{i-1} = c_{ij}\lambda_j + c_{ij-1}\beta_{j-1} \ (i, j = 1, 2, \ldots, m);$$

or $$c_{ij}\lambda_i + c_{i-1j}\beta_{i-1} = c_{ij}\lambda_j + c_{ij-1}\beta_{j-1} \ (i, j = 1, 2, \ldots, m),$$

since C is symmetric.

Just as in Ch. V, § 2, this gives

$$c_{ij} = 0 \text{ when } \lambda_i \neq \lambda_j,$$

and when $\lambda_i = \lambda_j$,

$$\left.\begin{array}{lll} c_{i-1j} = c_{ij-1} & \text{when} & \beta_{i-1} = 1,\ \beta_{j-1} = 1 \\ c_{ij-1} = 0 & \text{when} & \beta_{i-1} = 0,\ \beta_{j-1} = 1 \\ c_{i-1j} = 0 & \text{when} & \beta_{i-1} = 1,\ \beta_{j-1} = 0 \end{array}\right\} \ldots\ldots\ldots\ldots \text{(i)}$$

We have also $c_{ij} = c_{ji}$.*

These equations (i) may be handled in the same way as equations (iii) of Ch. V, § 2.

* This was not the case in Ch. V, § 2.

For instance, if N has the matrix (iv) of Ch. V, § 2, C will have a matrix of the type

$$\begin{vmatrix} 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & a & b & c & 0 & 0 & f \\ 0 & a & b & c & d & 0 & f & g \\ a & b & c & d & e & f & g & h \\ 0 & 0 & 0 & 0 & f & 0 & 0 & p \\ 0 & 0 & 0 & f & g & 0 & p & q \\ 0 & 0 & f & g & h & p & q & r \end{vmatrix}; \qquad \text{..................(ii)}$$

or if N has the matrix (vi) of Ch. V, § 2, C will have a matrix of the type

$$\begin{vmatrix} 0 & 0 & a & 0 & 0 & d & 0 & 0 \\ 0 & a & b & 0 & d & e & 0 & g \\ a & b & c & d & e & f & g & h \\ 0 & 0 & d & 0 & 0 & p & 0 & 0 \\ 0 & d & e & 0 & p & q & 0 & i \\ d & e & f & p & q & r & i & j \\ 0 & 0 & g & 0 & 0 & i & 0 & s \\ 0 & g & h & 0 & i & j & s & t \end{vmatrix} \qquad \text{..................(iii)}$$

From these examples the general form of C will be clear. The reader will notice the diagonal arrangement of equal elements in the matrix, and the positions of the zeros.

As in Ch. V, § 2, if N is expressed as the direct product of constituents $N_1, N_2, N_3, \ldots$, whose characteristic-determinants are $(\alpha-\lambda)^a$, $(\beta-\lambda)^b$, $(\gamma-\lambda)^c$, ..., where no two of $\alpha, \beta, \gamma, \ldots$ are equal; then C is the direct product of a substitution on the variables affected by N_1, a substitution on the variables affected by N_2, a substitution on the variables affected by N_3, ...

Corollary I.

A substitution with invariant-factors

$(\lambda-\alpha)^{a_1}, (\lambda-\alpha)^{a_2}, (\lambda-\alpha)^{a_3}, \ldots,$ where $a_1 \geq a_2 \geq a_3 \geq \ldots,$

$(\lambda-\beta)^{b_1}, (\lambda-\beta)^{b_2}, (\lambda-\beta)^{b_3}, \ldots,$ where $b_1 \geq b_2 \geq b_3 \geq \ldots,$

. . . .,

can be expressed as the product of two symmetric substitutions in a k-ply infinite number of ways where

$$k = (a_1+2a_2+3a_3+4a_4+\ldots)+(b_1+2b_2+3b_3+4b_4+\ldots)+\ldots.$$

The proof is as in Ch. V, § 2. The only difference is that the matrix of C in the present section is symmetric, while the corresponding matrix in Ch. V, § 2, is not.

Corollary II.

The determinant of C factorizes into simple factors.

For instance, the determinant (ii) is $\pm a^6 p^3$, and the determinant (iii) is

$$\pm \begin{vmatrix} a & d \\ d & p \end{vmatrix}^3 \times s^2.$$

In fact, if we write the rows and columns of (iii) which now stand in the order 1, 2, 3, 4, 5, 6, 7, 8 respectively in the orders 1, 3, 5, 2, 4, 6, 7, 8 and 5, 3, 1, 6, 4, 2, 7, 8, (iii) becomes

$$\begin{vmatrix} a & d & 0 & 0 & 0 & 0 & 0 & 0 \\ d & p & 0 & 0 & 0 & 0 & 0 & 0 \\ b & e & a & d & 0 & 0 & 0 & g \\ e & q & d & p & 0 & 0 & 0 & i \\ c & f & b & e & a & d & g & h \\ f & r & e & q & d & p & i & j \\ g & i & 0 & 0 & 0 & 0 & 0 & s \\ h & j & g & i & 0 & 0 & s & t \end{vmatrix}.$$

A similar process applies in general.

Corollary III.*

If K and $AK=L$ are symmetric, we can find a substitution R such that $RKR'=L$, where $R^2=A$ and R depends on A only (not on K).

Using the notation of § 2, we first prove that, if the result is true for the canonical substitution $N = PAP^{-1}$, it is true for A.

In fact, suppose NC and $C = PKP'$ are symmetric, while $DCD'=NC$, and $D^2 = N$ depends on N only (not on C). Then if $R = P^{-1}DP$,

$$RKR' = P^{-1}DP \,.\, P^{-1}CP'^{-1} \,.\, P'D'P'^{-1} = P^{-1}DCD'P'^{-1}$$
$$= P^{-1}NCP'^{-1} = P^{-1} \,.\, PAP^{-1} \,.\, PKP' \,.\, P'^{-1} = AK = L,$$

while $$R^2 = P^{-1}D^2P = P^{-1}NP = A,$$

and R depends solely on N.

We now show how D may be found.

Take at first for D the most general substitution permutable with every substitution permutable with N (Ch. V, § 3).

* This Corollary will not be required before Ch. IX. Similar theorems when K and AK are both alternate or both Hermitian are given in § 8, Ex. 6, 7, and § 11, Ex. 1, 2.

The reader will at once verify that, if C and NC are symmetric, then DC and NDC are symmetric.

For instance, if C has the matrix (iii) of § 4, and D is

$$\left.\begin{array}{l} x_1'=\mathrm{a}x_1+\mathrm{b}x_2+\mathrm{c}x_3,\ x_2'=\mathrm{a}x_2+\mathrm{b}x_3,\ x_3'=\mathrm{a}x_3 \\ x_4'=\mathrm{a}x_4+\mathrm{b}x_5+\mathrm{c}x_6,\ x_5'=\mathrm{a}x_5+\mathrm{b}x_6,\ x_6'=\mathrm{a}x_6 \\ x_7'=\mathrm{a}x_7+\mathrm{b}x_8,\ x_8'=\mathrm{a}x_8 \end{array}\right\},$$

the matrix of DC is also of the type (iii); and so on in general.

We therefore have $DC = C'D' = CD'$.

Now equate corresponding coefficients in D^2 and N, and we get equations which can evidently be satisfied by a proper choice of the coefficients of D. For instance, in the example just taken, where N has the matrix (vi) of Ch. V, § 2, we have

$$(\alpha+t)^{\frac{1}{2}} \equiv \mathrm{a}+\mathrm{b}t+\mathrm{c}t^2+\ldots.$$

We now have $D^2 = N$.

Therefore $DC = CD'$ gives $NC = D^2C = DCD'$; which was to be proved.

Ex. 1. Prove that the symmetric substitutions C such that NC is symmetric where $N \equiv (x_2,\ x_3,\ \ldots,\ x_m,\ e_1x_1+e_2x_2+\ldots+e_mx_m)$ are m-ply infinite in number.

[$c_{ij-1}+e_jc_{im} = c_{j\,i-1}+e_ic_{jm}$. Put in turn $j = 1, 2, \ldots, m$, and then $i = 1, 2, \ldots, m$, and we determine the coefficients of C in terms of $c_{1m}, c_{2m}, \ldots, c_{mm}$ which are arbitrary.]

Ex. 2. Let $A \equiv (3x+y-2z,\ 2x+3y-3z,\ 4x+2y-3z)$

and $K \equiv (-4x-2y+3z,\ -2x+y,\ 3x-z)$.

Find R so that $R^2 = A$,

$$RKR' = AK \equiv (-4x-4y+5z,\ -4x+y+z,\ 5x+y-3z).$$

[In Corollary III

$$N \equiv (x+y,\ y+z,\ z),\ P \equiv (x-y,\ 2x-y+z,\ 2x-2y+z),$$

$$P^{-1} \equiv (x+y-z,\ y-z,\ -2x+z),\ C = PKP' \equiv (z,\ y,\ x).$$

Then $D^2 = N$, where $D \equiv (x+\frac{1}{2}y-\frac{1}{8}z,\ y+\frac{1}{2}z,\ z)$; and

$$R = P^{-1}DP \equiv (\tfrac{9}{4}x+\tfrac{1}{2}y-\tfrac{9}{8}z,\ \tfrac{3}{2}x+2y-\tfrac{7}{4}z,\ \tfrac{5}{2}x+y-\tfrac{5}{4}z).]$$

If $K_1 \equiv (-4x-4y+5z,\ -4x+y+z,\ 5x+y-3z)$, verify that

$$RK_1R' = AK_1 \equiv (-6y+5z,\ -6x+y+2z,\ 5x+2y-4z).$$

[This illustrates the fact that R depends solely on A, not on K. In this case $PK_1P' \equiv (z,\ y+z,\ x+y)$.]

§ 5. Transformation of one Symmetric Substitution into another.

We assume for the present the following theorem, which will be proved in § 6.

If N is a given canonical substitution which is the direct product of substitutions of the type

$$x_1' = \alpha x_1 + x_2, \ \ldots, \ x'_{r-1} = \alpha x_{r-1} + x_r, \ x_r' = \alpha x_r,$$

and C is the direct product of substitutions of the type

$$x_1' = x_r, \ \ldots, \ x'_{r-1} = x_2, \ x_r' = x_1,$$

and F is any symmetric substitution such that NF is symmetric, then we can find a substitution D permutable with N such that $DFD' = C$.

From this we deduce the following theorems :—

If a substitution A is transformable into the canonical substitution N, and AK and K are symmetric, we can find a substitution P such that $P^{-1}NP = A$ and $PKP' = C$.

Let R be any substitution such that $R^{-1}NR = A$, and let $RKR' = F$. Then, by § 2, NF is symmetric. Hence we can find D permutable with N such that $DFD' = C$. Take $P = DR$.

Then $\quad P^{-1}NP = R^{-1}D^{-1} . N . DR = R^{-1}NR = A$;

and $\qquad PKP' = DR . K . R'D' = DFD' = C.$

Again,

If two substitutions A and B have the same invariant-factors, while AK, BK, and K are all symmetric, we can find a substitution S such that $S^{-1}AS = B$ and $SKS' = K$.

We have just shown that we can find substitutions P and Q such that $A = P^{-1}NP$, $B = Q^{-1}NQ$, $PKP' = C$, $QKQ' = C$, where N is the canonical form of A or B.

Put $S = P^{-1}Q$.

Then $\quad S^{-1}AS = Q^{-1}P . A . P^{-1}Q = Q^{-1}NQ = B$;

and $\qquad SKS' = P^{-1}Q . K . Q'P'^{-1} = P^{-1}CP'^{-1} = K.$

If $K = E$, A and B are symmetric; and $SS' = E$, so that S is orthogonal. Hence

Any given symmetric substitution can be transformed into another given symmetric substitution with the same invariant-factors by an orthogonal substitution.

Ex. Transform $A \equiv (-\frac{7}{25}x+\frac{24}{25}y,\ \frac{24}{25}x+\frac{7}{25}y)$
into $B \equiv (\frac{7}{25}x+\frac{24}{25}y,\ \frac{24}{25}x-\frac{7}{25}y)$
by an orthogonal substitution.

[$A = P^{-1}NP$, $B = Q^{-1}NQ$, where $N \equiv (x,\ -y)$,

$$P \equiv (\tfrac{3}{5}x-\tfrac{4}{5}y,\ \tfrac{4}{5}x+\tfrac{3}{5}y),\quad Q \equiv (\tfrac{4}{5}x-\tfrac{3}{5}y,\ \tfrac{3}{5}x+\tfrac{4}{5}y).$$

Therefore A is transformed into B by the orthogonal substitution

$$P^{-1}Q \equiv (\tfrac{24}{25}x+\tfrac{7}{25}y,\ -\tfrac{7}{25}x+\tfrac{24}{25}y).]$$

§ 6.

We now prove the first theorem of § 5.

We notice that, if D is *any* substitution whatever permutable with N, and NF is symmetric, then

$$N.DFD' = D.NF.D'$$

is symmetric (Ch. I, § 10). Hence DFD' is of the same type as F, namely the type exemplified in (ii) and (iii) of § 4. We want to show that we can choose D so that DFD' is C. This is the same thing as proving that we can choose D so that, when we replace

$$x_t \text{ by } d_{t1}x_1+d_{t2}x_2+\ldots+d_{tm}x_m \quad (t = 1, 2, \ldots, m),$$

N is unaltered while the quadratic form $\Sigma f_{ij}x_ix_j$ is transformed into $\Sigma c_{ij}x_ix_j$, i.e. into the sum of quadratic forms of the type

$$x_1x_r+x_2x_{r-1}+x_3x_{r-2}+\ldots+x_rx_1.$$

We shall adopt a step-by-step process whereby we employ a succession of transformations, each of which leaves N unaltered, but alters the quadratic form $\Sigma f_{ij}x_ix_j$ into a simpler form, till we finally arrive at $\Sigma c_{ij}x_ix_j$.

As pointed out at the end of § 4, if N is the direct product of constituents $N_1, N_2, N_3, \ldots$ whose characteristic-determinants are $(\alpha-\lambda)^a$, $(\beta-\lambda)^b$, $(\gamma-\lambda)^c$, ..., where no two of $\alpha, \beta, \gamma, \ldots$ are equal, $\Sigma f_{ij}x_ix_j$ is the sum of a quadratic form on the variables affected by N_1, a quadratic form on the variables affected by N_2, a quadratic form on the variables affected by N_3, ...

Suppose N_1 is the direct product of $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \ldots$, where $\mathcal{N}_1$ is the direct product of constituents $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots$, each of the type

$$x_1' = \alpha x_1+x_2,\ \ldots,\ x'_{u-1} = \alpha x_{u-1}+x_u,\ x_u' = \alpha x_u,$$

and therefore each of the *same* degree u, while $\mathcal{N}_2$ is the direct product of constituents of the same degree v, $\mathcal{N}_3$ is the

direct product of constituents of the same degree $w, \ldots$, where no two of $u, v, w, \ldots$ are equal; and similarly for $N_2, N_3, \ldots$.

Then, firstly, we show that $\Sigma f_{ij} x_i x_j$ may be transformed without altering N into the sum of a quadratic form on the variables affected by $\mathcal{N}_1$, a quadratic form on the variables affected by $\mathcal{N}_2$, a quadratic form on the variables affected by $\mathcal{N}_3, \ldots$.

Secondly, we show that $\Sigma f_{ij} x_i x_j$ may then be transformed without altering N into the sum of a quadratic form on the variables affected by $\mathcal{M}_1$, a quadratic form on the variables affected by $\mathcal{M}_2$, a quadratic form on the variables affected by $\mathcal{M}_3, \ldots$.

Thirdly, we show that the theorem is true for each of the substitutions such as $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots$; and the proof is then completed.

In the following we therefore first take N as of the type N_1, then of the type $\mathcal{N}_1$, and finally of the type $\mathcal{M}_1$.

(1) Consider first of all a case such as that in which N is

$$x_1' = \alpha x_1 + x_2,\ x_2' = \alpha x_2 + x_3,\ x_3' = \alpha x_3;$$
$$x_4' = \alpha x_4 + x_5,\ x_5' = \alpha x_5 + x_6,\ x_6' = \alpha x_6;$$
$$x_7' = \alpha x_7 + x_8,\ x_8' = \alpha x_8.$$

Here N is the direct product of two substitutions of degree 3 (one on x_1, x_2, x_3 and the other on x_4, x_5, x_6) and of another substitution (on x_7, x_8) of degree less than 3.* Then $\Sigma f_{ij} x_i x_j$ has a matrix of the type (iii) of § 4.

$$\left.\begin{array}{lll} \text{Put for } x_7 & & x_7 + kx_2 + k'x_3 + lx_5 + l'x_6 \\ \text{and for } x_8 & & x_8 \qquad + kx_3 \qquad\quad + lx_6 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

By Ch. V this transformation does not alter N.

It transforms $\Sigma f_{ij} x_i x_j$ into a form with matrix

$$\left|\begin{array}{cccccccc} \circ & \circ & A & \circ & \circ & D & \circ & \circ \\ \circ & A & B & \circ & D & E & \circ & G \\ A & B & C & D & E & F & G & H \\ \circ & \circ & D & \circ & \circ & P & \circ & \circ \\ \circ & D & E & \circ & P & Q & \circ & I \\ D & E & F & P & Q & R & I & J \\ \circ & \circ & G & \circ & \circ & I & \circ & S \\ \circ & G & H & \circ & I & J & S & T \end{array}\right|,$$

where

$$G = ks + g,\quad H = k's + kt + h,\quad I = ls + i,\quad J = l's + lt + j.$$

* N is of the type N_1, being the direct product of

$\mathcal{N}_1 \equiv (\alpha x_1 + x_2,\ \alpha x_2 + x_3,\ \alpha x_3,\ \alpha x_4 + x_5,\ \alpha x_5 + x_6,\ \alpha x_6)$ and $\mathcal{N}_2 \equiv (\alpha x_7 + x_8,\ \alpha x_8)$.

Now since the determinant of $\Sigma f_{ij}x_ix_j$ does not vanish, $s \neq 0$. Hence we can choose k, k', l, l' so that G, H, I, J all vanish; and $\Sigma f_{ij}x_ix_j$ becomes a quadratic form on $x_1, x_2, x_3, x_4, x_5, x_6$, *plus* a quadratic form on x_7, x_8.

A similar process holds in general. For instance, if N were

$$x_1' = \alpha x_1 + x_2,\ x_2' = \alpha x_2 + x_3,\ x_3' = \alpha x_3;$$
$$x_4' = \alpha x_4 + x_5,\ x_5' = \alpha x_5;\ \ x_6' = \alpha x_6 + x_7,\ x_7' = \alpha x_7,$$

which is the direct product of substitutions of degree 3, 2, 2, we should put

$$\left.\begin{array}{llll} \text{for } x_4 & x_4 + kx_2 + k'x_3, & \text{for } x_6 & x_6 + lx_2 + l'x_3 \\ \text{for } x_5 & x_5 \qquad + kx_3, & \text{for } x_7 & x_7 \qquad + lx_3 \end{array}\right\},$$

and proceed as before, leaving N unaltered and transforming $\Sigma f_{ij}x_ix_j$ into a quadratic form on x_1, x_2, x_3, *plus* a quadratic form on x_4, x_5, x_6, x_7 by suitable choice of k, k', l, l'.

(2) We can now confine ourselves to the case in which N is the direct product of constituents each of which is of the same degree.

Suppose, for example, N is

$$x_1' = \alpha x_1 + x_2,\ x_2' = \alpha x_2 + x_3,\ x_3' = \alpha x_3;$$
$$x_4' = \alpha x_4 + x_5,\ x_5' = \alpha x_5 + x_6,\ x_6' = \alpha x_6,$$

so that N is the direct product of two constituents, each of degree 3.*

$\Sigma f_{ij}x_ix_j$ will have a matrix of the type

$$\begin{vmatrix} 0 & 0 & r_{11} & 0 & 0 & r_{12} \\ 0 & r_{11} & s_{11} & 0 & r_{12} & s_{12} \\ r_{11} & s_{11} & t_{11} & r_{12} & s_{12} & t_{12} \\ 0 & 0 & r_{21} & 0 & 0 & r_{22} \\ 0 & r_{21} & s_{21} & 0 & r_{22} & s_{22} \\ r_{21} & s_{21} & t_{21} & r_{22} & s_{22} & t_{22} \end{vmatrix}, \quad \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

where $$r_{12} = r_{21},\ s_{12} = s_{21},\ t_{12} = t_{21}.$$

$$\left.\begin{array}{ll} \text{For } x_1 \text{ put } l_{11}x_1 + l_{12}x_4, & \text{for } x_4 \text{ put } l_{21}x_1 + l_{22}x_4 \\ \text{for } x_2 \text{ put } l_{11}x_2 + l_{12}x_5, & \text{for } x_5 \text{ put } l_{21}x_2 + l_{22}x_5 \\ \text{for } x_3 \text{ put } l_{11}x_3 + l_{12}x_6, & \text{for } x_6 \text{ put } l_{21}x_3 + l_{22}x_6 \end{array}\right\} \ldots\text{(iii)}$$

This will not alter N.

Choose the l's so that

$$r_{11}(l_{11}x_1 + l_{12}x_2)^2 + 2r_{12}(l_{11}x_1 + l_{12}x_2)(l_{21}x_1 + l_{22}x_2)$$
$$+ r_{22}(l_{21}x_1 + l_{22}x_2)^2 \equiv x_1^2 + x_2^2.$$

* N is of the type $\mathfrak{N}_1$, being the direct product of $\mathcal{M}_1 \equiv (\alpha x_1 + x_2,\ \alpha x_2 + x_3,\ \alpha x_3)$ and $\mathcal{M}_2 \equiv (\alpha x_4 + x_5,\ \alpha x_5 + x_6,\ \alpha x_6)$.

This is possible, for the determinant (ii) is

$$\begin{vmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{vmatrix}^3, \quad \text{so that} \quad \begin{vmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{vmatrix} \neq 0.*$$

Then $\Sigma f_{ij} x_i x_j$ becomes a quadratic form with a matrix of the type

$$\begin{vmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & S_{11} & 0 & 0 & S_{12} \\ 1 & S_{11} & T_{11} & 0 & S_{12} & T_{12} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & S_{21} & 0 & 1 & S_{22} \\ 0 & S_{21} & T_{21} & 1 & S_{22} & T_{22} \end{vmatrix},$$

where $$S_{12} = S_{21}, \quad T_{12} = T_{21}.$$

Put now for x_1 $\quad x_1 + kx_5 + lx_6$
and for x_2 $\quad x_2 \quad + kx_6$ $\Big\}$.

This will not alter N, and, exactly as in (1), we can choose k and l to make $\Sigma f_{ij} x_i x_j$ reduce to a quadratic form which is the sum of a quadratic form on x_1, x_2, x_3 and a quadratic form on x_4, x_5, x_6.

(3) We can now confine ourselves to the case in which N is of the form

$$x_1' = \alpha x_1 + x_2, \ldots, x'_{r-1} = \alpha x_{r-1} + x_r, \; x_r' = \alpha x_r.\dagger$$

$\Sigma f_{ij} x_i x_j$ is of the form

$$k_1 \phi_1(x, x) + k_2 \phi_2(x, x) + \ldots + k_r \phi_r(x, x),$$

where
$$\begin{aligned} \phi_1(x, y) &\equiv x_1 y_r + x_2 y_{r-1} + \ldots + x_{r-1} y_2 + x_r y_1, \\ \phi_2(x, y) &\equiv x_2 y_r + x_3 y_{r-1} + \ldots + x_{r-1} y_3 + x_r y_2, \\ \phi_3(x, y) &\equiv x_3 y_r + x_4 y_{r-1} + \ldots + x_{r-1} y_4 + x_r y_3, \\ &\cdots\cdots\cdots\cdots, \\ \phi_r(x, y) &\equiv x_r y_r, \end{aligned}$$

and the k's are constants.

If in $\phi_1(x, y)$ we put

$$\left.\begin{aligned} a_r x_1 + a_{r-1} x_2 + a_{r-2} x_3 + \ldots + a_1 x_r &\text{ for } x_1 \\ a_r x_2 + a_{r-1} x_3 + \ldots + a_2 x_r &\text{ for } x_2 \\ a_r x_3 + \ldots + a_3 x_r &\text{ for } x_3 \\ \cdots\cdots\cdots\cdots\cdots &\cdots \\ a_r x_r &\text{ for } x_r \\ b_r y_1 + b_{r-1} y_2 + b_{r-2} y_3 + \ldots + b_1 y_r &\text{ for } y_1 \\ b_r y_2 + b_{r-1} y_3 + \ldots + b_2 y_r &\text{ for } y_2 \\ b_r y_3 + \ldots + b_3 y_r &\text{ for } y_3 \\ \cdots\cdots\cdots\cdots\cdots &\cdots \\ b_r y_r &\text{ for } y_r \end{aligned}\right\} \quad \ldots\ldots\ldots(\text{iv})$$

* Compare § 4, Corollary II.

† N is of the type $\mathcal{M}_1$.

we obtain

$$(a_r x_1 + a_{r-1} x_2 + \dots + a_1 x_r) b_r y_r + (a_r x_2 + \dots + a_2 x_r)(b_r y_{r-1} + b_{r-1} y_r) \\ + \dots + a_r x_r (b_r y_1 + b_{r-1} y_2 + \dots + b_1 y_r)$$

$$\equiv \phi_r(a, b) \,.\, \phi_1(x, y) + \phi_{r-1}(a, b) \,.\, \phi_2(x, y) \\ + \dots + \phi_2(a, b) \,.\, \phi_{r-1}(x, y) + \phi_1(a, b) \,.\, \phi_r(x, y).$$

Similarly, $\phi_2(x, y)$ becomes

$$\phi_r(a, b)\, \phi_2(x, y) + \phi_{r-1}(a, b)\, \phi_3(x, y) + \dots + \phi_2(a, b) \,.\, \phi_r(x, y),$$

and so for $\phi_3(x, y)$, $\phi_4(x, y)$, ...

Putting now

$$b_1 = a_1,\ b_2 = a_2,\ \dots,\ y_1 = x_1,\ y_2 = x_2,\ \dots$$

(iv) becomes a substitution, leaving N unaltered and transforming

$$k_1 \phi_1(x, x) + k_2 \phi_2(x, x) + \dots + k_r \phi_r(x, x)$$

into $$K_1 \phi_1(x, x) + K_2 \phi_2(x, x) + \dots + K_r \phi_r(x, x),$$

where

$$K_1 = k_1 \phi_r(a, a),\ K_2 = k_1 \phi_{r-1}(a, a) + k_2 \phi_r(a, a), \\ K_3 = k_1 \phi_{r-2}(a, a) + k_2 \phi_{r-1}(a, a) + k_3 \phi_r(a, a). \dots.$$

We want to show that we can choose $a_r, a_{r-1}, a_{r-2}, \dots$ to satisfy $K_1 = 1$, $K_2 = K_3 = \dots = 0$.

This demands

$$k_1 a_r^2 = 1,\ 2k_1 a_r a_{r-1} + k_2 a_r^2 = 0, \\ k_1(2a_r a_{r-2} + a^2_{r-1}) + 2k_2 a_r a_{r-1} + k_3 a_r^2 = 0, \dots.$$

The a's can evidently be chosen to satisfy these equations, provided $k_1 \neq 0$.

But if $k_1 = 0$, the determinant of $\Sigma f_{ij} x_i x_j$ would vanish, which is impossible.

The investigation is now completed and the theorem proved.

Ex. 1. Prove that the orthogonal substitutions permutable with a given symmetric substitution A of degree r having a single invariant-factor are two in number.

[By a suitable change of variables we reduce the problem to that of finding the substitutions permutable with

$$x_1' = \alpha x_1 + x_2,\ \dots,\ x'_{r-1} = \alpha x_{r-1} + x_r,\ x_r' = \alpha x_r,$$

and having $$x_1 x_r + x_2 x_{r-1} + \dots + x_{r-1} x_2 + x_r x_1$$

as an invariant. For an orthogonal substitution has

$$x_1^2 + x_2^2 + \dots + x_r^2$$

as invariant, and this was transformed into

$$x_1 x_r + x_2 x_{r-1} + \dots + x_r x_1.$$

The substitutions required are those of the form (iv) of § 6, which have the invariant $x_1 x_r + x_2 x_{r-1} + \dots$. They will have this invariant if $\phi_r(a, a) = 1$ and $\phi_{r-1}(a, a) = \phi_{r-2}(a, a) = \dots = 0$. These equations give $a_r = \pm 1$, $a_{r-1} = a_{r-2} = \dots = 0$.]

Ex. 2. Discuss the case when A has two invariant-factors, or any number of invariant-factors.*

Ex. 3. In how many ways can a given symmetric substitution be transformed into another given symmetric substitution by an orthogonal substitution?

§ 7. Invariant-factors of an Alternate Substitution.

If A is an *alternate* substitution, $A = SA'$ where S is the similarity substitution

$$x_1' = -x_1,\ x_2' = -x_2,\ \dots,\ x_m' = -x_m.$$

Therefore, by Ch. II, § 3, corresponding to any invariant-factor $(\lambda - \alpha)^r$ of A' we have an invariant-factor $(\lambda + \alpha)^r$ of SA' or A. But A and A' have the same invariant-factors. Hence

The invariant-factors of an alternate substitution are either powers of λ, *or else occur in pairs of the type* $(\lambda - \alpha)^r$, $(\lambda + \alpha)^r$.†

We confine ourselves now to the case of alternate substitutions whose determinant does not vanish. The degree of such an alternate substitution is even,‡ and its invariant-factors occur in pairs of the type $(\lambda - \alpha)^r$, $(\lambda + \alpha)^r$. It possesses many properties analogous to those of a symmetric substitution. We state these properties, leaving the verification to the reader in most cases.

Suppose $P^{-1}NP = A$ and $PKP' = C$. *Then if* AK *is alternate, so is* NC; *and, conversely, if* NC *is alternate, so is* AK.

The proof is as in § 2.

For example, take $K = E$ and $C = E$. Then we have:—
'The transform of an alternate substitution by an orthogonal substitution is alternate,' as in Ch. I, § 10.

Again; still taking $K = E$:—

To find an alternate substitution with given pairs of invariant-factors.

* Cf. Hilton, *Annals of Mathematics* (1914).
† See also Ch. II, § 3, Ex. 3.
‡ For a skew-symmetric determinant of odd order vanishes.

It is sufficient to take one pair of invariant-factors $(\lambda-\alpha)^r$ and $(\lambda+\alpha)^r$.

Then if N is

$$\left.\begin{array}{l} x_1'=\alpha x_1 \quad +x_2,\ \ldots,\ x'_{r-1}=\alpha x_{r-1} \quad +x_r,\ x_r'=\alpha x_r \\ y_1'=-\alpha y_1+y_2,\ \ldots,\ y'_{r-1}=-\alpha y_{r-1}+y_r,\ y_r'=-\alpha y_r \end{array}\right\},\ \ldots \text{(i)}$$

and C is

$$\left.\begin{array}{l} x_1'=y_r, \qquad x_2'=-y_{r-1} \qquad \ldots,\ x_r'=(-1)^{r+1}y_1 \\ y_1'=(-1)^{r+1}x_r,\ y_2'=(-1)^r x_{r-1},\ \ldots,\ y_r'=x_1 \end{array}\right\},\ \ldots \text{(ii)}$$

NC is alternate and C symmetric.

Then the required substitution is $P^{-1}NP$, where P is chosen so that $C=PP'$.

Ex. 1. Find an alternate substitution with invariant-factors $(\lambda-\alpha)^2$, $(\lambda+\alpha)^2$.

[$2x_1x_4-2x_2x_3\equiv\left(\frac{x_1+x_4}{\sqrt{2}}\right)^2+\left(\frac{x_2-x_3}{\sqrt{2}}\right)^2+\left(\frac{ix_2+ix_3}{\sqrt{2}}\right)^2+\left(\frac{ix_1-ix_4}{\sqrt{2}}\right)^2.$

Therefore the substitution $C\equiv(x_4,\ -x_3,\ -x_2,\ x_1)$ is PP', where

$$P\equiv\left(\frac{x_1+x_4}{\sqrt{2}},\quad \frac{x_2-x_3}{\sqrt{2}},\quad \frac{ix_2+ix_3}{\sqrt{2}},\quad \frac{ix_1-ix_4}{\sqrt{2}}\right).$$

And $\qquad N\equiv(\alpha x_1+x_2,\ \alpha x_2,\ -\alpha x_3+x_4,\ -\alpha x_4)$

is transformed by P into the required alternate substitution with matrix

$$\left|\begin{array}{cccc} 0 & \frac{1}{2} & -\frac{i}{2} & -i\alpha \\ -\frac{1}{2} & 0 & -i\alpha & -\frac{i}{2} \\ \frac{i}{2} & i\alpha & 0 & -\frac{1}{2} \\ i\alpha & \frac{i}{2} & \frac{1}{2} & 0 \end{array}\right|.]$$

Ex. 2. Find an alternate substitution with invariant-factors $(\lambda-\alpha)^3$, $(\lambda+\alpha)^3$.

§ 8. The Product of Two Alternate Substitutions.

Suppose now that K is alternate. Then if $AK=L$ is alternate, NC is alternate, and so is $C=PKP'$.

Each alternate substitution C, such that NC is also alternate, gives one method of expressing A as the product of two alternate substitutions L and K^{-1} (i. e. $P'C^{-1}P$).

We consider then the conditions that NC should be alternate when C is alternate. As in § 4, we get, since NC is alternate,

$$c_{ji}\lambda_i + c_{ji-1}\beta_{i-1} = -c_{ij}\lambda_j - c_{ij-1}\beta_{j-1}\ (i, j = 1, 2, \ldots, m),$$

or

$$c_{ij}\lambda_i + c_{i-1j}\beta_{i-1} = c_{ij}\lambda_j + c_{ij-1}\beta_{j-1}\ (i, j = 1, 2, \ldots, m),$$

since C is alternate.

But these are the same conditions as in § 4 except that now we have $c_{ij} = -c_{ji}$ instead of $c_{ij} = c_{ji}$.

Suppose, for instance, that N is

$$x_1' = \alpha x_1 + x_2,\ x_2' = \alpha x_2 + x_3,\ x_3' = \alpha x_3;$$
$$x_4' = \alpha x_4 + x_5,\ x_5' = \alpha x_5 + x_6,\ x_6' = \alpha x_6.$$

Then the matrix of C is of the type

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & d & e & f \\ 0 & 0 & -d & 0 & 0 & 0 \\ 0 & -d & -e & 0 & 0 & 0 \\ -d & -e & -f & 0 & 0 & 0 \end{vmatrix},$$

which may be compared with the matrices in § 4.

Applying Corollary II of § 4, we see that, since the determinant of C is not zero, and a skew-symmetric determinant of odd order vanishes:—

A substitution can be expressed as the product of two alternate substitutions if and only if its invariant-factors occur in pairs of the form $(\lambda-\alpha)^r$, $(\lambda-\alpha)^r$.

Ex. 1. In how many ways can a given substitution be expressed as the product of two alternate substitutions?

[See Corollary I of § 4.]

Ex. 2. Express

$$A \equiv (x-y+2z-2w,\ -3y-2w,\ -2x+2y-3z+3w,\ 2y+w)$$

as the product of two alternate substitutions.

[$P^{-1}NP = A$, where $N \equiv (-x+y,\ -y,\ -z+w,\ -w)$, and $P^{-1} \equiv (3x+2z,\ 2x+y+2z,\ x+z,\ y+w)$.

Then $K \equiv P^{-1}CP'^{-1} \equiv (-2y-3w,\ 2x+z,\ -y-2w,\ 3x+2z)$, where $C \equiv (-w,\ -z,\ y,\ x)$. Hence A is the product of the two alternate substitutions $AK \equiv (w,\ z-w,\ -y,\ -x+y)$ and

$$K^{-1} \equiv (2y-w,\ -2x+3z,\ -3y+2w,\ x-2z).]$$

Ex. 3. A substitution of the second degree is the product of two alternate substitutions, if and only if it is a similarity.

Ex. 4. If F and NF are alternate, we can find a substitution D permutable with N so that $DFD' = C$, where N and C are the direct products of

$$\left.\begin{array}{llll} x_1' = \alpha x_1 + x_2, & \ldots, & x'_{r-1} = \alpha x_{r-1} + x_r, & x_r' = \alpha x_r \\ y_1' = \alpha y_1 + y_2, & \ldots, & y'_{r-1} = \alpha y_{r-1} + y_r, & y_r' = \alpha y_r \end{array}\right\}$$

and
$$\left.\begin{array}{llll} x_1' = -y_r, & \ldots, & x'_{r-1} = -y_2, & x_r' = -y_1 \\ y_1' = x_r, & \ldots, & y'_{r-1} = x_2, & y_r' = x_1 \end{array}\right\}$$

respectively.

Ex. 5. If AK and K are alternate, we can find a substitution P so that $PAP^{-1} = N$ and $PKP' = C$, where C is the substitution defined in Ex. 4.

If AK, BK, and K are alternate, and A, B have the same invariant-factors, we can find a substitution S such that

$$S^{-1}AS = B \text{ and } SKS' = K.$$

Ex. 6. If K and AK are alternate, we can find a substitution R depending only on A (not on K) such that $R^2 = A$ and $RKR' = AK$.

[The method is that of § 4, Corollary III.]

Ex. 7. If $A \equiv 2x - y + w,\ y - z + w,\ -x + y,\ -x + y - z + w)$ and $K \equiv (-y + 2z - 2w,\ x - z + 2w,\ -2x + y - w,\ 2x - 2y + z)$, find R such that $R^2 = A$ and

$$RKR' = AK \equiv (-y + 3z - 3w,\ x - 2z + 3w,\ -3x + 2y - 2w,\ 3x - 3y + 2z).$$

[If $P \equiv (x - w,\ 2x - 2y - z - w,\ x - 3y - z + w,\ x - y - z)$ and $P^{-1} \equiv (3x - 2y + z + w,\ x - y + w,\ 2x - y + z - w,\ 2x - 2y + z + w)$,

then $$PAP^{-1} = N \equiv (x + y,\ y,\ z + w,\ w)$$

and $$C = PKP' \equiv (w,\ z,\ -y,\ -x).$$

If $D \equiv (x + \frac{1}{2}y,\ y,\ z + \frac{1}{2}w,\ w)$, $D^2 = N$ and $DCD' = NC$.

Hence, as in § 4, Corollary III,

if $$R = P^{-1}DP \equiv (\tfrac{3}{2}x - \tfrac{1}{2}y + \tfrac{1}{2}w,\ y - \tfrac{1}{2}z + \tfrac{1}{2}w,\ -\tfrac{1}{2}x + \tfrac{1}{2}y + \tfrac{1}{2}z,\ -\tfrac{1}{2}x + \tfrac{1}{2}y - \tfrac{1}{2}z + w),$$

$R^2 = A$ and $RKR' = AK$.]

If $K_1 \equiv (-y + 3z - 3w, x - 2z + 3w, -3x + 2y - 2w, 3x - 3y + 2z)$, verify that

$$RK_1R' = AK_1 \equiv (-y + 4z - 4w,\ x - 3z + 4w,\ -4x + 3y - 3w,\ 4x - 4y + 3z).$$

[$PK_1P' \equiv (w,\ z + w,\ -y,\ -x - y)$.]

§ 9. The Product of an Alternate and a Symmetric Substitution.

Suppose now that K is *symmetric*. Then if $AK = L$ is alternate, NC is alternate and $C = PKP'$ is symmetric.

Each substitution C, such that NC is alternate and C symmetric, gives one method of expressing a given substitution whose invariant-factors occur in pairs of the type $(\lambda-\alpha)^r$, $(\lambda+\alpha)^r$ as the product of an alternate substitution L and a symmetric substitution K^{-1}.

If N is the direct product of substitutions of the type (i) of § 7, we may take C as the direct product of substitutions of the type (ii) of § 7.

If the invariant-factors of A do not occur in pairs of the type $(\lambda-\alpha)^r$, $(\lambda+\alpha)^r$, A cannot be expressed as the product LK^{-1} where L is alternate and K^{-1} symmetric.

For let Q be a substitution such that $QK^{-1}Q' = E$ (§ 1). Then the invariant-factors of LK^{-1} are the same as those of

$$Q'^{-1}LK^{-1}Q' = Q'^{-1}LQ^{-1}\,.\,QK^{-1}Q' = Q'^{-1}LQ^{-1}.$$

But $Q'^{-1}LQ^{-1}$ is alternate, since L is. Hence the invariant-factors occur in pairs of the type $(\lambda-\alpha)^r$, $(\lambda+\alpha)^r$ by § 7.

Therefore

A substitution can be expressed as the product of an alternate and a symmetric substitution, if and only if its invariant-factors occur in pairs of the type $(\lambda-\alpha)^r$, $(\lambda+\alpha)^r$.

To find every possible symmetric substitution such as C making NC alternate, we notice that, as in § 4, we have

$$c_{ji}\lambda_i+c_{ji-1}\beta_{i-1} = -c_{ij}\lambda_j-c_{ij-1}\beta_{j-1}\ (i, j = 1, 2, \ldots, m);$$

or $$c_{ij}\lambda_i+c_{i-1j}\beta_{i-1} = -c_{ij}\lambda_j-c_{ij-1}\beta_{j-1}\ (i, j = 1, 2, \ldots, m),$$

since C is symmetric.

This leads, as in § 4, to

$$c_{ij} = 0 \quad\text{when}\quad \lambda_i+\lambda_j \neq 0,$$

and when $\lambda_i = -\lambda_j$

$$\left.\begin{array}{lll} c_{i-1j} = -c_{ij-1} & \text{when} & \beta_{i-1} = 1,\ \beta_{j-1} = 1 \\ c_{ij-1} = 0 & \text{when} & \beta_{i-1} = 0,\ \beta_{j-1} = 1 \\ c_{i-1j} = 0 & \text{when} & \beta_{i-1} = 1,\ \beta_{j-1} = 0 \end{array}\right\}.$$

We have also $c_{ij} = c_{ji}$.

Suppose, for instance, that N is the substitution

$$x_1' = \alpha x_1+x_2,\quad x_2' = \alpha x_2+x_3,\quad x_3' = \alpha x_3;$$
$$y_1' = -\alpha y_1+y_2,\ y_2' = -\alpha y_2+y_3,\ y_3' = -\alpha y_3;$$
$$x_4' = \alpha x_4+x_5,\quad x_5' = \alpha x_5+x_6,\quad x_6' = \alpha x_6;$$
$$y_4' = -\alpha y_4+y_5,\ y_5' = -\alpha y_5+y_6,\ y_6' = -\alpha y_6;$$
$$\left.\begin{array}{ll} x_7' = \alpha x_7+x_8, & x_8' = \alpha x_8 \\ y_7' = -\alpha y_7+y_8, & y_8' = -\alpha y_8 \end{array}\right\}\ldots\ (\mathrm{i})$$

Then C has a symmetric matrix of order 16, of which the first 8 elements of the first 8 rows and the last 8 elements of the last 8 rows are zero, while the last 8 elements of the first 8 rows are

$$\begin{bmatrix} 0 & 0 & a & 0 & 0 & d & 0 & 0 \\ 0 & -a & -b & 0 & -d & -e & 0 & -g \\ a & b & c & d & e & f & g & m \\ 0 & 0 & h & 0 & 0 & p & 0 & 0 \\ 0 & -h & -k & 0 & -p & -q & 0 & -i \\ h & k & n & p & q & r & i & j \\ 0 & 0 & -u & 0 & 0 & -l & 0 & -s \\ 0 & u & v & 0 & l & w & s & t \end{bmatrix} \quad \ldots\ldots\ldots \text{(ii)}$$

This should be compared with (iii) of § 4.

The reader will easily verify that the determinant of the substitution C formed in this manner would vanish if the invariant-factors of N did not occur in pairs of the type $(\lambda-\alpha)^r$, $(\lambda+\alpha)^r$, as was to be expected from the previous reasoning.

We assume now a theorem proved in § 10.

If N is a given canonical substitution which is the direct product of substitutions of the type

$$\left.\begin{aligned} x_1' &= \alpha x_1 + x_2, \quad \ldots, x'_{r-1} = \alpha x_{r-1} + x_r, \quad x_r' = \alpha x_r \\ y_1' &= -\alpha y_1 + y_2, \ldots, y'_{r-1} = -\alpha y_{r-1} + y_r, y_r' = -\alpha y_r \end{aligned}\right\},$$

and C is the direct product of substitutions of the type

$$\left.\begin{aligned} &x_1' = y_r, \qquad x_2' = -y_{r-1}, \qquad x_3' = y_{r-2}, \\ &y_1' = (-1)^{r-1}x_r, y_2' = (-1)^{r-2}x_{r-1}, y_3' = (-1)^{r-3}x_{r-2}, \\ &\qquad\qquad\qquad\qquad \ldots, x_r' = (-1)^{r-1}y_1 \\ &\qquad\qquad\qquad\qquad \ldots, y_r' = x_1 \end{aligned}\right\},$$

and F is any symmetric substitution such that NF is alternate; then we can find a substitution D permutable with N such that $DFD' = C$.

As in § 5, we deduce

If a substitution A is transformable into the canonical substitution N, and AK is alternate and K symmetric, we can find a substitution P such that $P^{-1}NP = A$ and $PKP' = C$.

Again :—

If two substitutions A and B have the same invariant-factors, while AK and BK are both alternate and K is

symmetric, we can find a substitution S such that $S^{-1}AS = B$ and $SKS' = K$.

Taking $K = E$ we have

Any given alternate substitution can be transformed into another given alternate substitution with the same invariant-factors by an orthogonal substitution.

Ex. 1. Express
$$A \equiv (3x+3y+10z+10w,\ 4x-y+4z+6w,\ -2x+2y+z-w,\ -2y-4z-3w)$$
as the product of an alternate and a symmetric substitution.

[$P^{-1}NP = A$
if $P \equiv (2x+y+z+w,\ 2x+2z-w,\ -x+y-z+w,\ -y-w)$, and $N \equiv (x+y,\ y,\ -z+w,\ -w)$. Take C as $(w,\ -z,\ -y,\ x)$. Then
$$K = P^{-1}CP'^{-1} = (-w,\ z+w,\ y+4z+2w,\ -x+y+2z-2w).$$
Hence A is the product of the alternate substitution
$$AK \equiv (2y+4z+3w,\ -2x-3z-4w,\ -4x+3y-4w,\ -3x+4y+4z),$$
and the symmetric substitution
$$K^{-1} \equiv (2x-2y+z-w,\ -2x-4y+z,\ x+y,\ -x).]$$

Ex. 2. Express $(34x-24y,\ 48x-34y)$ as the product of an alternate and a symmetric substitution.

§ 10.

We now prove the theorem left over from § 9. The process is very similar to that of § 6, and will therefore be given only in outline. We have to show that we can choose D so that when we replace x_t by
$$d_{t1}x_1 + d_{t2}x_2 + \ldots + d_{tm}x_m \quad (t = 1, 2, \ldots, m)$$
N is unaltered and $\Sigma f_{ij}x_ix_j$ is transformed into $\Sigma c_{ij}x_ix_j$, i.e. into the sum of quadratic form of the type
$$2\,(y_rx_1 - y_{r-1}x_2 + y_{r-2}x_3 - y_{r-3}x_4 + \ldots),$$
N being of the type stated in the enunciation of the theorem.

(1) Take such a case as that in which N is (i) of § 9. Then the matrix of $\Sigma f_{ij}x_ix_j$ is of the type described in § 9 in which the first 8 elements of the first 8 rows and the last 8 elements of the last 8 rows are zero, and the remaining portions of the matrix are of the type (ii) of § 9.

Now apply the substitution (i) of § 6. Choosing k, k', l, l' properly we reduce $\Sigma f_{ij}x_ix_j$ to one similar to that just

described, but with u, v, l, w zero. Now apply a similar transformation to y_7 and y_8, and we can make g, m, i, j also zero.

(2) We may now confine ourselves to the case in which N is the direct product of constituents each of the same degree.

Suppose, for example, N is

$$x_1' = \alpha x_1 + x_2, \quad x_2' = \alpha x_2 + x_3, \quad x_3' = \alpha x_3\,;$$
$$y_1' = -\alpha y_1 + y_2,\ y_2' = -\alpha y_2 + y_3,\ y_3' = -\alpha y_3\,;$$
$$\left.\begin{matrix} x_4' = \alpha x_4 + x_5, & x_5' = \alpha x_5 + x_6, & x_6' = \alpha x_6 \\ y_4' = -\alpha y_4 + y_5, & y_5' = -\alpha y_5 + y_6, & y_6' = -\alpha y_6 \end{matrix}\right\}.$$

$\Sigma f_{ij} x_i x_j$ will have a symmetric matrix of order 12 in which the first 6 elements of the first 6 rows and the last 6 elements of the last 6 rows are zero, and the last 6 elements of the first 6 rows (and the first 6 elements of the last 6 rows) are

$$\begin{bmatrix} 0 & 0 & r_{11} & 0 & 0 & r_{12} \\ 0 & -r_{11} & -s_{11} & 0 & -r_{12} & -s_{12} \\ r_{11} & s_{11} & t_{11} & r_{12} & s_{12} & t_{12} \\ 0 & 0 & r_{21} & 0 & 0 & r_{22} \\ 0 & -r_{21} & -s_{21} & 0 & -r_{22} & -s_{22} \\ r_{21} & s_{21} & t_{21} & r_{22} & s_{22} & t_{22} \end{bmatrix} \ldots\ldots\ldots\ldots \text{(i)}$$

Apply the transformation (iii) of § 6. Choose the l's in this transformation so that the substitutions with matrices

$$\begin{vmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{vmatrix}$$

are inverses of each other. Then (i) is replaced by an array of the form

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -S_{11} & 0 & 0 & -S_{12} \\ 1 & S_{11} & T_{11} & 0 & S_{12} & T_{12} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -S_{21} & 0 & -1 & -S_{22} \\ 0 & S_{21} & T_{21} & 1 & S_{22} & T_{22} \end{bmatrix},$$

and now we can proceed as in (1).

(3) Lastly, take N as (i) of § 7. Then

$$\begin{aligned} \Sigma f_{ij} x_i x_j \equiv 2k_1 (y_r x_1 - y_{r-1} x_2 + y_{r-2} x_3 - \ldots) \\ + 2k_2 (y_r x_2 - y_{r-1} x_3 + y_{r-2} x_4 - \ldots) \\ + 2k_3 (y_r x_3 - y_{r-1} x_4 + y_{r-2} x_5 - \ldots) + \ldots . \end{aligned}$$

Now apply the transformation (iv) of § 6, taking

$$b_r = 1,\ b_{r-1} = b_{r-2} = b_{r-3} = \ldots = 0.$$

Then choose $a_r, a_{r-1}, a_{r-2}, \ldots$ to make $\Sigma f_{ij} x_i x_j$ reduce to

$$2\,(y_r x_1 - y_{r-1} x_2 + y_{r-2} x_3 - \ldots).$$

Ex. If $F_1(x, y) \equiv 2\,(y_r x_1 - y_{r-1} x_2 + y_{r-2} x_3 - y_{r-3} x_4 + \ldots)$,

$F_2(x, y) \equiv 2\,(y_r x_2 - y_{r-1} x_3 + y_{r-2} x_4 - y_{r-3} x_5 + \ldots)$,

. . . .

then $F_1(x, y)$ becomes

$$F_r(a, b) F_1(x, y) + F_{r-1}(a, b) F_2(x, y) + \ldots + F_1(a, b),\ F_r(x, y),$$

when we transform it by (iv) of § 6.

§ 11. The Product of Two Hermitian Substitutions.

The following results are similar to those proved in §§ 2–10 and are established similarly.

Suppose $P^{-1}NP = A$ and $PK\bar{P}' = C$. Then, if AK is Hermitian, so is NC; and, conversely, if NC is Hermitian, so is AK.

A given substitution A can be expressed as the product of two Hermitian substitutions, if and only if its complex invariant-factors occur in pairs of the type $(\lambda - \alpha)^r$, $(\lambda - \bar{\alpha})^r$.

Firstly, A cannot be so expressed unless its invariant-factors are thus paired. In fact, if K and $AK = L$ are Hermitian, and Q is a substitution such that $QK^{-1}\bar{Q}'$ is a real multiplication (§ 1), then A has the same invariant-factors as

$$\bar{Q}'^{-1} L K^{-1} \bar{Q}' = \bar{Q}'^{-1} L Q^{-1} \,.\, Q K^{-1} \bar{Q}',$$

which is the product of a Hermitian substitution and a real multiplication. Now use Ch. II, § 3, Ex. 2.*

Again, take N as the canonical substitution into which A is transformable. Then if the invariant-factors of A are so paired, N will be the direct product of substitutions of the type

$$\left.\begin{aligned} x_1' &= \alpha x_1 + x_2, \ \ldots, x'_{r-1} = \alpha x_{r-1} + x_r,\ x_r' = \alpha x_r \\ y_1' &= \bar{\alpha} y_1 + y_2, \ \ldots, y'_{r-1} = \bar{\alpha} y_{r-1} + y_r,\ y_r' = \bar{\alpha} y_r \\ X_1' &= \beta X_1 + X_2, \ldots, X'_{s-1} = \beta X_{s-1} + X_s,\ X_s' = \beta X_s \end{aligned}\right\},$$

where β is real and α unreal.

* Or we may establish the result by combining Ch. I, § 6, Ex. 13, and Ch. II, § 2, Ex. 4, without introducing Q.

We may take C as the direct product of substitutions of the type

$$\left.\begin{array}{l} x_1' = y_r, \ \ldots, x'_{r-1} = y_2, \ x_r' = y_1 \\ y_1' = x_r, \ \ldots, y'_{r-1} = x_2, \ y_r' = x_1 \\ X_1' = X_s, \ldots, X'_{s-1} = X_2, X_s' = X_1 \end{array}\right\}.$$

Then C and NC are Hermitian, and therefore AK and K are Hermitian.

Moreover, if NF and F are Hermitian, we can transform $\Sigma f_{ij}\bar{x}_i x_j$ into the sum of Hermitian forms of the type

$$\pm\,(\bar{y}_r x_1 + \bar{y}_{r-1}x_2 + \ldots + \bar{y}_1 x_r + \bar{x}_r y_1 + \bar{x}_{r-1}y_2 + \ldots + \bar{x}_1 y_r)$$

and $$\pm\,(X_1\bar{X}_s + X_2\bar{X}_{s-1} + \ldots + X_{s-1}\bar{X}_2 + X_s\bar{X}_1)$$

without altering N.

The proof is as in § 10, noting that the equations (i) of § 4 now become

$$c_{ij} = 0 \quad \text{when} \quad \lambda_i \neq \bar{\lambda}_j,$$

and $$c_{i-1j}\beta_{i-1} = \bar{c}_{ij-1}\beta_{j-1} \quad \text{when} \quad \lambda_i = \bar{\lambda}_j$$

together with $c_{ij} = \bar{c}_{ji}$.

Ex. 1. If K and AK are Hermitian. we can find a substitution R depending only on A (not on K) such that $R^2 = A$ and $RA\bar{R}' = AK$.

[The method is that of § 4, Corollary III.]

Ex. 2. If $A \equiv (x+2y, \ -x-y)$ and $K \equiv (-x-y, \ -x)$, find R such that $R^2 = A$ and $RK\bar{R}' = AK \equiv (-y, \ -x-2y)$.

[If $$P \equiv \left(ix+y, \ -\frac{1+i}{2}x - \frac{1+i}{2}y\right)$$

and $$P^{-1} \equiv \left(-\frac{1+i}{2}x - y, \ \frac{1+i}{2}x + iy\right),$$

then $PAP^{-1} = N \equiv (ix, \ -iy)$, and $C = PK\bar{P}' \equiv (y, \ x)$.

If $D \equiv \left(\frac{1+i}{\sqrt{2}}x, \ \frac{1-i}{\sqrt{2}}y\right)$, $D^2 = N$ and $DC\bar{D}' = NC$.

Hence, as in § 4, Corollary III,

if $$R = P^{-1}DP \equiv \left(\sqrt{2}x + \sqrt{2}y, \ -\frac{x}{\sqrt{2}}\right),$$

$R^2 = A$ and $RK\bar{R}' = AK$.]

§ 12. The Product of a Hermitian Substitution and a Real Multiplication.

We can deduce properties of a substitution A which is the product of a Hermitian substitution and a real multiplication K.

Suppose that of the coefficients of K, k are positive and $m-k$ negative. There will be no loss of generality in supposing $k \geq \frac{1}{2}m$.

The Hermitian form $\Sigma k_{ij}\bar{x}_i x_j$ is transformable into the form $\Sigma f_{ij}\bar{x}_i x_j$, where $F = PK\bar{P}'$. Hence, when $\Sigma f_{ij}\bar{x}_i x_j$ is reduced to standard form, as in Ch. III, § 2, it becomes

$$\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \ldots + \xi_k\bar{\xi}_k - \xi_{k+1}\bar{\xi}_{k+1} - \ldots - \xi_m\bar{\xi}_m.$$

Now $\Sigma f_{ij}\bar{x}_i x_j$ was also reduced in § 11 to the sum of forms of the type

$$\pm\,(\bar{y}_r x_1 + \bar{y}_{r-1}x_2 + \ldots + \bar{x}_1 y_r)$$

and
$$\pm\,(X_1\bar{X}_s + X_2\bar{X}_{s-1} + \ldots + X_s\bar{X}_1).$$

But each of these is reducible by the method of Ch. III, § 6, to the type

$$\eta_1\bar{\eta}_1 - \eta_2\bar{\eta}_2 + \eta_3\bar{\eta}_3 - \eta_4\bar{\eta}_4 + \ldots,$$

as is at once seen by use of the identity

$$2\,(x\bar{y} + \bar{x}y) \equiv (x+y)\,(\bar{x}+\bar{y}) - (x-y)\,(\bar{x}-\bar{y}).$$

Hence :—

'A cannot have more than $2\,(m-k)$ unreal characteristic-roots. If exactly $2k-m$ characteristic-roots are real, the corresponding invariant-factors are linear.'

Again :—

'A cannot have more than $m-k$ invariant-factors which are not linear. If A has exactly $m-k$ such non-linear invariant-factors, they are of degree 2 or 3. If they are all of degree 3, every characteristic-root of A is real.'

'If A has exactly $3k-2m$ linear invariant-factors, all its characteristic-roots are real, and the non-linear invariant-factors are of degree 3.'

Ex. 1. Enunciate the theorems at the end of § 12, for the case $2k < m$.

Ex. 2. Obtain, by means of § 12, properties of a 'quasi-Hermitian substitution', i.e. one of the type

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m,$$

where $a_{ij} = \bar{a}_{ji}$ if i and j are both $\leq k$ or both $> k$; and otherwise $a_{ij} = -\bar{a}_{ji}$.

[Take K as $(x_1, \ldots, x_k, -x_{k+1}, \ldots, -x_m)$.]

Ex. 3. Find quasi-Hermitian substitutions with invariant-factors (i) $(\lambda-\alpha)^2$, α real; (ii) $(\lambda-\alpha)$, $(\lambda-\bar{\alpha})$; (iii) $(\lambda-\alpha)^2$, $(\lambda-\bar{\alpha})^2$.

[The substitutions with matrices

$$\begin{vmatrix} 1+\alpha & -1 \\ 1 & -1+\alpha \end{vmatrix}, \quad \begin{vmatrix} \frac{1}{2}(\alpha+\bar{\alpha}) & \frac{1}{2}(\alpha-\bar{\alpha}) \\ \frac{1}{2}(\alpha-\bar{\alpha}) & \frac{1}{2}(\alpha+\bar{\alpha}) \end{vmatrix},$$

$$\begin{vmatrix} \frac{1}{2}(\alpha+\bar{\alpha}) & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}(\alpha-\bar{\alpha}) \\ \frac{1}{2} & \frac{1}{2}(\alpha+\bar{\alpha}) & \frac{1}{2}(\alpha-\bar{\alpha}) & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}(\alpha-\bar{\alpha}) & \frac{1}{2}(\alpha+\bar{\alpha}) & \frac{1}{2} \\ \frac{1}{2}(\alpha-\bar{\alpha}) & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}(\alpha+\bar{\alpha}) \end{vmatrix}.$$

Take N as a canonical substitution with the given invariant-factors, K as in Ex. 2, and C as in § 11.

Choose P so that $PK\bar{P}' = C$ (§ 1, Ex. 7). Then A is $P^{-1}NP$.]

Ex. 4. In how many ways can a given substitution be expressed as the product of two Hermitian substitutions?

[See § 4, Corollary I.]

CHAPTER VII

INVARIANTS OF THE SECOND DEGREE

§ 1. Quadratic Invariants of any Substitution.

In Ch. I, § 11, we defined an (absolute) invariant of the substitution A

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \dots + a_{tm}x_m \quad (t = 1, 2, \dots, m)$$

and proved that if $PAP^{-1} = B$ and $f(x_1, x_2, \dots, x_m)$ is an invariant of A, then

$$f(p_{11}x_1 + p_{12}x_2 + \dots + p_{1m}x_m,\ p_{21}x_1 + p_{22}x_2 + \dots + p_{2m}x_m,\ \dots,\ p_{m1}x_1 + p_{m2}x_2 + \dots + p_{mm}x_m)$$

is an invariant of B.

If

$$f(x_1, x_2, \dots, x_m) \equiv \Sigma c_{ij}x_ix_j \ (i, j = 1, 2, \dots, m),\ c_{ij} = c_{ji}$$

is a *quadratic invariant*, we can give an alternative proof of this fact as follows:—

Suppose C to be the substitution

$$x_t' = c_{t1}x_1 + c_{t2}x_2 + \dots + c_{tm}x_m,$$

then, since $\Sigma c_{ij}x_ix_j$ is an invariant of A, $ACA' = C$ by Ch. III, § 1.

Then it is required to prove that $\Sigma d_{ij}x_ix_j$ is an invariant of B, where

$$\Sigma c_{ij}(p_{i1}x_1 + p_{i2}x_2 + \dots + p_{im}x_m)(p_{j1}x_1 + p_{j2}x_2 + \dots + p_{jm}x_m) \equiv \Sigma d_{ij}x_ix_j,$$

or $D = PCP'$.

Hence we must prove $BDB' = D$.

But

$$BDB' = PAP^{-1}.PCP'.P'^{-1}A'P' = P.ACA'.P' = PCP' = D,$$

which establishes the result.

We proceed to find all the quadratic invariants of A. By the above it is sufficient to find the quadratic invariants of any substitution N into which A may be transformed; for then the quadratic invariants of A can be readily deduced.

We might take for N the well-known canonical form which is the direct product of substitutions of the type

$$x_1' = \alpha x_1 + x_2,\ x_2' = \alpha x_2 + x_3,\ \ldots,\ x'_{r-1} = \alpha x_{r-1} + x_r,\ x_r' = \alpha x_r.$$

It will save labour, however, to take N as the direct product of substitutions of the type

$$x_1' = \alpha x_1 + \alpha x_2,\ x_2' = \alpha x_2 + \alpha x_3,\ \ldots,\ x'_{r-1} = \alpha x_{r-1} + \alpha x_r,\ x_r' = \alpha x_r,$$

into which the previous substitution is at once transformed by writing

$$\alpha x_2 \text{ for } x_2,\quad \alpha^2 x_3 \text{ for } x_3,\ \ldots,\quad \alpha^{r-1} x_r \text{ for } x_r,$$

i.e. the former substitution is transformed into the latter by

$$x_1' = x_1,\ x_2' = \alpha^{-1} x_2,\ x_3' = \alpha^{-2} x_3,\ \ldots,\ x_r' = \alpha^{-r+1} x_r.$$

The substitutions permutable with this new canonical form N are the same as those permutable with the usual form and found in Ch. V, § 2.

Ex. 1. If $$I \equiv e_1 x_1 + e_2 x_2 + \ldots + e_m x_m$$
is any *relative linear invariant* of A, $(e_1, e_2, \ldots, e_m)$ is a pole of A'; and when we operate with A on I, I becomes λI, where λ is the corresponding characteristic-root of A'.

Ex. 2. The number of independent absolute linear invariants of A is equal to the number of invariant-factors of A of the type $(\lambda - 1)^a$.

§ 2. Quadratic Invariants of a Canonical Substitution.

The most general quadratic invariant of the canonical substitution N of § 1 is evidently obtained by taking any pair of the constituent substitutions of which N is the direct product, finding every quadratic invariant of this pair, and then adding all the invariants so obtained. The pair chosen may be distinct, or may be the same constituent reckoned twice over.

First suppose that the pair is distinct and is

$$\left.\begin{aligned} x_1' &= \alpha x_1 + \alpha x_2,\ \ldots,\ x'_{r-1} = \alpha x_{r-1} + \alpha x_r,\ x_r' = \alpha x_r \\ y_1' &= \beta y_1 + \beta y_2,\ \ldots,\ y'_{s-1} = \beta y_{s-1} + \beta y_s,\ y_s' = \beta y_s \end{aligned}\right\} \ldots\ (\text{i})$$

Take first of all $r = s$.

Let an invariant of this pair be

$$\Sigma\, e_{ij} y_i x_j \quad (i, j = 1, 2, \ldots, r).$$

Then $$\Sigma\, e_{ij} (y_i' x_j' - y_i x_j) \equiv 0,$$

where x_j', y_i' are given in terms of $x_1, \ldots, x_r, y_1, \ldots, y_r$ by (i).

Equating to zero the coefficients of

$$y_1x_1, \ldots, y_1x_m;\ y_2x_1, \ldots, y_2x_m;\ \ldots;\ y_mx_1, \ldots\, y_mx_m$$

we see that the following quantities vanish:—

$$\begin{array}{ll}
(\alpha\beta-1)\,e_{11}, & (\alpha\beta-1)\,e_{12}+e_{11}, \\
(\alpha\beta-1)\,e_{21}+e_{11}, & (\alpha\beta-1)\,e_{22}+e_{21}+e_{12}+e_{11}, \\
(\alpha\beta-1)\,e_{31}+e_{21}, & (\alpha\beta-1)\,e_{32}+e_{31}+e_{22}+e_{21}, \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
(\alpha\beta-1)\,e_{r1}+e_{r-1\,1}, & (\alpha\beta-1)\,e_{r2}+e_{r1}+e_{r-1\,2}+e_{r-1\,1},
\end{array}$$

$$\begin{array}{ll}
(\alpha\beta-1)\,e_{13}+e_{12}, & \ldots, \\
(\alpha\beta-1)\,e_{23}+e_{22}+e_{13}+e_{12}, & \ldots, \\
(\alpha\beta-1)\,e_{33}+e_{32}+e_{23}+e_{22}, & \ldots, \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\
(\alpha\beta-1)\,e_{r3}+e_{r2}+e_{r-1\,3}+e_{r-1\,2}, & \ldots;
\end{array}$$

$$\left.\begin{array}{l}
(\alpha\beta-1)\,e_{1\,r}+e_{1\,r-1} \\
(\alpha\beta-1)\,e_{2\,r}+e_{2\,r-1}+e_{1\,r}+e_{1\,r-1} \\
(\alpha\beta-1)\,e_{3\,r}+e_{3\,r-1}+e_{2\,r}+e_{2\,r-1} \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
(\alpha\beta-1)\,e_{rr}+e_{rr-1}+e_{r-1\,r}+e_{r-1\,r-1}
\end{array}\right\} \ldots\ldots\text{(ii)}$$

A glance at these quantities shows that $e_{ij}=0$ for each value of i and j unless $\alpha\beta=1$.

If $\alpha\beta=1$, we have at once

$$e_{11}=e_{12}=\ldots=e_{1\,r-1}=0,\ e_{21}=e_{22}=\ldots=e_{2\,r-2}=0, \ldots, e_{r-1\,1}=0,$$

together with $\frac{1}{2}r(r-1)$ relations of the type

$$\begin{array}{lll}
e_{r1}+e_{r-1\,2}=0, & e_{r-1\,2}+e_{r-2\,3}=0, & \ldots, \\
e_{r2}+e_{r-1\,2}+e_{r-1\,3}=0, & e_{r-1\,3}+e_{r-2\,3}+e_{r-2\,4}=0, & \ldots, \\
e_{r3}+e_{r-1\,3}+e_{r-1\,4}=0, & e_{r-1\,4}+e_{r-2\,4}+e_{r-2\,5}=0, & \ldots, \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot
\end{array}$$

$$\left.\begin{array}{l}
e_{2\,r-1}+e_{1\,r}=0 \\
e_{3\,r-1}+e_{2\,r-1}+e_{2\,r}=0 \\
e_{4\,r-1}+e_{3\,r-1}+e_{3\,r}=0 \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}\right\},$$

i. e. in the matrix

$$\left|\begin{array}{cccccc}
e_{11} & e_{12} & \cdot\ \cdot\ \cdot & e_{1\,r-1} & e_{1\,r} \\
e_{21} & e_{22} & \cdot\ \cdot\ \cdot & e_{2\,r-1} & e_{2\,r} \\
\cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot \\
e_{r-1\,1} & e_{r-1\,2} & \cdot\ \cdot\ \cdot & e_{r-1\,r-1} & e_{r-1\,r} \\
e_{r1} & e_{r2} & \cdot\ \cdot\ \cdot & e_{rr-1} & e_{rr}
\end{array}\right|, \ldots\ldots\ldots\text{(iii)}$$

all the elements to the left and above the diagonal from e_{r1} to e_{1r} are zero; the sum of two adjacent elements in this diagonal is zero; and the sum of two adjacent elements in a parallel diagonal * together with the element above one of these elements and to the left of the other is zero.

It follows at once that $e_{r1}, e_{r2}, \ldots, e_{rr}$ may be chosen arbitrarily, and then every element of the matrix (iii) is uniquely determined. Hence there are r independent invariants of (i).

By suitable choice of $e_{r1}, e_{r2}, \ldots, e_{rr}$ we can throw the invariants into a great variety of forms. We shall obtain here the most symmetrical shape. It is necessary to distinguish the cases in which r is odd and r is even. First take r even, i.e. $r = 2n$.

Then choose

$$e_{r1} = 1,\ e_{r2} = {}^{n-1}C_1,\ e_{r3} = {}^{n-1}C_2,\ \ldots,\ e_{rn} = {}^{n-1}C_{n-1},$$
$$e_{r\,n+1} = \ldots = e_{rr} = 0.$$

The matrix (iii) becomes, when each element is multiplied by $(-1)^{n-1}$, of the type

$$\left|\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & {}^5C_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -{}^4C_3 & {}^5C_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & {}^3C_2 & -{}^4C_2 & {}^5C_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -{}^2C_1 & {}^3C_1 & -{}^4C_1 & {}^5C_1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & {}^2C_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -{}^3C_2 & -{}^3C_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & {}^4C_3 & {}^4C_2 & {}^4C_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -{}^5C_4 & -{}^5C_3 & -{}^5C_2 & -{}^5C_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right|, \quad \text{(iv)}$$

where for illustration we take $n = 6$.

Next take r odd, i.e. $r = 2n+1$.

Then choose

$$e_{r1} = 1,\ e_{r2} = {}^{n-1}C_{n-2} + \tfrac{1}{2},\ e_{r3} = {}^{n-1}C_{n-3} + \tfrac{1}{2}\,{}^{n-1}C_{n-2},\ \ldots,$$
$$e_{rn} = 1 + \tfrac{1}{2}{}^{n-1}C_1,\ e_{r\,n+1} = \tfrac{1}{2},\ e_{r\,n+2} = \ldots = e_{rr} = 0.$$

* Those running from e_{r2} to e_{2r}, e_{r3} to e_{3r}, ..., $e_{r\,r-1}$ to $e_{r-1\,r}$.

The matrix (iii) becomes, when each element is multiplied by $(-1)^n$, of the type

$$\begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -({}^4C_3+\frac{1}{2}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & ({}^3C_2+\frac{1}{2}) & -({}^4C_2+\frac{1}{2}{}^4C_3) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -({}^2C_1+\frac{1}{2}) & ({}^3C_1+\frac{1}{2}{}^3C_2) & -({}^4C_1+\frac{1}{2}{}^4C_2) \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & (1+\frac{1}{2}) & -(1+\frac{1}{2}{}^2C_1) & (1+\frac{1}{2}{}^3C_1) & -(1+\frac{1}{2}{}^4C_1) \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
0 & 0 & 0 & 0 & -1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & (1+\frac{1}{2}) & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -({}^2C_1+\frac{1}{2}) & -(1+\frac{1}{2}{}^2C_1) & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & ({}^3C_2+\frac{1}{2}) & ({}^3C_1+\frac{1}{2}{}^3C_2) & (1+\frac{1}{2}{}^3C_1) & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
-1 & -({}^4C_3+\frac{1}{2}) & -({}^4C_2+\frac{1}{2}{}^4C_3) & -({}^4C_1+\frac{1}{2}{}^4C_2) & -(1+\frac{1}{2}{}^4C_1) & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0
\end{vmatrix} \quad (v)$$

where for illustration we take $n = 5$.

The reader will notice that the matrix is symmetric when r is odd, and skew-symmetric when r is even.

We shall denote the invariant whose matrix we have obtained by $f_1(x, y)$.

The substitution

$$\left.\begin{array}{l} x_2' = \alpha x_2 + \alpha x_3, x_3' = \alpha x_3 + \alpha x_4, \ldots, x'_{r-1} = \alpha x_{r-1} + \alpha x_r, x_r' = \alpha x_r \\ y_2' = \beta y_2 + \beta y_3, y_3' = \beta y_3 + \beta y_4, \ldots, y'_{r-1} = \beta y_{r-1} + \beta y_r, y_r' = \beta y_r \end{array}\right\}$$

will have an exactly analogous invariant which we shall denote by $f_2(x, y)$.

The substitution

$$\left.\begin{array}{l} x_3' = \alpha x_3 + \alpha x_4, x_4' = \alpha x_4 + \alpha x_5, \ldots, x'_{r-1} = \alpha x_{r-1} + \alpha x_r, x_r' = \alpha x_r \\ y_3' = \beta y_3 + \beta y_4, y_4' = \beta y_4 + \beta y_5, \ldots, y'_{r-1} = \beta y_{r-1} + \beta y_r, y_r' = \beta y_r \end{array}\right\}$$

has an analogous invariant which we shall denote by $f_3(x, y)$, and so on.

The quantities

$$f_1(x, y), f_2(x, y), \ldots, f_r(x, y)$$

are evidently invariants of (i). They are the r independent invariants of (i) whose existence we proved before.

We have

$$f_r(x, y) \equiv x_r y_r ; \; f_{r-1}(x, y) \equiv x_{r-1} y_r - x_r y_{r-1} ;$$

$$f_{r-2}(x, y) \equiv x_{r-2} y_r - x_{r-1} y_{r-1} + x_r y_{r-2} + \tfrac{1}{2} x_{r-1} y_r - \tfrac{1}{2} x_r y_{r-1} ;$$

$$f_{r-3}(x, y) \equiv x_{r-3} y_r - x_{r-2} y_{r-1} + x_{r-1} y_{r-2} - x_r y_{r-3} + x_{r-2} y_r - x_r y_{r-2} ;$$

and so on.

The term of lowest weight* involving y_r has unit coefficient in each invariant.

Now we suppose $r \neq s$, e.g. $s > r$.

The argument is similar to that employed in the case of $s = r$, and may be left to the reader. The invariants are those obtained in the case of $s = r$, but with $y_{s-r+1}, y_{s-r+2}, \ldots, y_s$ written instead of $y_1, y_2, \ldots, y_r$.†

This might have been anticipated, for the required invariants of (i) are obviously invariants of

$$\left.\begin{array}{l} x_1' = \alpha x_1 + \alpha x_2, \qquad \ldots, x'_{r-1} = \alpha x_{r-1} + \alpha x_r, x_r' = \alpha x_r \\ y'_{s-r+1} = \beta y_{s-r+1} + \beta y_{s-r+2}, \ldots, y'_{s-1} = \beta y_{s-1} + \beta y_s, y_s' = \beta y_s \end{array}\right\}.$$

* The sum of the suffixes of x and y.

† There is no invariant unless $\alpha\beta = 1$.

We shall still denote the invariants by

$$f_1(x, y),\ f_2(x, y),\ \ldots,\ f_r(x, y).$$

Now take the case in which the pair of constituents (i) of N is identical, so that

$$\alpha = \beta,\ r = s;\ y_1 = x_1,\ y_2 = x_2,\ \ldots,\ y_r = x_r.$$

Any quadratic invariant of (i) will now be $\Sigma e_{ij} x_i x_j$, where $e_{ij} = e_{ji}$.

Equating to zero in $\Sigma e_{ij}(x_i' x_j' - x_i x_j)$ the coefficients of

$$x_1^2,\ x_1 x_2,\ \ldots,\ x_1 x_r;\ x_2^2,\ x_2 x_3,\ \ldots,\ x_2 x_r;\ \ldots;\ x_r^2,$$

we see that in the array (ii) (which should have been printed as a determinant, had space allowed) all quantities in the leading diagonal and to the right of and above this leading diagonal must vanish.

But the array (ii) is now symmetrical about its leading diagonal; and hence, as before, all elements of the array (ii) must vanish.

Hence, as before, there is no invariant unless $\alpha^2 = 1$; and if $\alpha^2 = 1$ the matrix (iii) takes the form illustrated in (iv) and (v). But if r is even, the matrix (iv) is skew-symmetric; and therefore $f_1(x, x)$, obtained from $f_1(x, y)$ by putting $y_1 = x_1,\ y_2 = x_2,\ \ldots,\ y_r = x_r$, vanishes identically if r is even.

Hence the required invariants are

$$f_r(x, x),\ f_{r-2}(x, x),\ f_{r-4}(x, x),\ \ldots,$$

$I[\frac{1}{2}(r+1)]$ in number.*

We have

$$f_r(x, x) = x_r^2,\ f_{r-2}(x, x) = 2x_{r-2}x_r - x^2_{r-1} + x_{r-1}x_r,$$
$$f_{r-4}(x, x) = 2x_{r-4}x_r - 2x_{r-3}x_{r-1} + x^2_{r-2} + 3x_{r-3}x_r - x_{r-2}x_{r-1} + x_{r-2}x_r,$$

and so on.

Return now to the canonical substitution N. Suppose it to be the direct product of substitutions on variables $x_1, x_2, \ldots$ with characteristic-root α ($\alpha^2 \neq 1$), of substitutions on variables $y_1, y_2, \ldots$ with characteristic-root α^{-1} (and so on for each characteristic-root whose square is not unity), and also of substitutions on variables $X_1, X_2, \ldots$ with characteristic-root $+1$, and of substitutions on variables $\xi_1, \xi_2, \ldots$ with characteristic-root -1.

Then the most general quadratic invariant of N will be, as explained at the beginning of the section, the sum of

* $I[x]$ means 'the integral part of x'.

a quadratic form on the variables X, a quadratic form on the variables ξ, a bilinear form on the variables x and y, and of similar bilinear forms for each characteristic-root whose square is not unity.

The determinant of the quadratic invariant of N will be the product of the determinants of these quadratic or bilinear forms, of which it is the sum.

Ex. 1. What invariant is obtained by taking

$$e_{r1} = 1,\ e_{r2} = e_{r3} = \dots = e_{rr} = 0 \text{ in § 2?}$$

[$y_r x_1 + y_{r-1}(-x_2 + x_3 - x_4 + x_5 - \dots)$
$+ y_{r-2}(x_3 - {}^2C_1 x_4 + {}^3C_1 x_5 - {}^4C_1 x_6 + \dots)$
$+ y_{r-3}(-x_4 + {}^3C_2 x_5 - {}^4C_2 x_6 + {}^5C_2 x_7 - \dots)$
$+ y_{r-4}(x_5 - {}^4C_3 x_6 + {}^5C_3 x_7 - {}^6C_3 x_8 + \dots) + \dots$.]

Ex. 2. Find the quadratic invariants of

$$x' = y,\ y' = e_1 x + e_2 y.$$

[If $e_1 = -1$, $a(x^2 - e_2 xy + y^2)$.
If $e_1 = 1$ and $e_2 = 0$, $a(x^2 + y^2) + 2hxy$.
If $e_2 \neq 0$ and $e_1 = 1 \mp e_2$, $a(e_1 x \pm y)^2$.]

Ex. 3. Find the independent quadratic invariants of

$$x_1' = x_2,\ x_2' = -\alpha^2 x_1 + 2\alpha x_2,\ y_1' = y_2,\ y_2' = -\beta^2 y_1 + 2\beta y_2,$$

where $\alpha\beta = 1$ and $\alpha^2 \neq 1$.

[$y_1 x_1 + y_2 x_2 - \beta y_1 x_2 - \alpha y_2 x_1$ and $\beta y_1 x_2 - \alpha y_2 x_1$.]

Ex. 4. There is no cubic invariant $\Sigma a_{ijk} x_i y_j z_k$ of

$$\left.\begin{array}{l} x_1' = \alpha x_1 + \alpha x_2,\ \dots,\ x'_{r-1} = \alpha x_{r-1} + \alpha x_r,\ x_r' = \alpha x_r \\ y_1' = \beta y_1 + \beta y_2,\ \dots,\ y'_{r-1} = \beta y_{r-1} + \beta y_r,\ y_r' = \beta y_r \\ z_1' = \gamma z_1 + \gamma z_2,\ \dots,\ z'_{r-1} = \gamma z_{r-1} + \gamma z_r,\ z_r' = \gamma z_r \end{array}\right\},$$

unless $\alpha\beta\gamma = 1$.

If $\alpha\beta\gamma = 1$, find the independent cubic invariants in the cases $r = 1, 2, 3$.

[$r = 1$ gives the invariant $x_1 y_1 z_1$.
$r = 2$ gives $x_2 y_2 z_2$ and $x_2(y_1 z_2 - y_2 z_1)$, $y_2(z_1 x_2 - z_2 x_1)$.*
$r = 3$ gives $x_3 y_3 z_3$; $x_3(y_2 z_3 - y_3 z_2)$, $y_3(z_2 x_3 - z_3 x_2)$;
and $x_1 y_2 z_3 - x_1 y_3 z_2 + x_3 y_1 z_2 - x_2 y_1 z_3 + x_2 y_3 z_1 - x_3 y_2 z_1$,
with $-x_3 y_2 z_2 + x_3 y_1 z_3 + x_3 y_3 z_1 + x_2 y_3 z_3$,
and two similar invariants.

Discuss the cases $\beta = \gamma$, $y = z$ and $\alpha = \beta = \gamma$, $x = y = z$.]

* $z_2(x_1 y_2 - x_2 y_1)$ is not independent of these.

Ex. 5. There is no quartic invariant $\Sigma a_{ijkl}\, x_i y_j z_k w_l$ of

$$\left.\begin{array}{l} x_1' = \alpha x_1 + \alpha x_2,\ \ldots,\ x'_{r-1} = \alpha x_{r-1} + \alpha x_r,\ x_r' = \alpha x_r \\ y_1' = \beta y_1 + \beta y_2,\ \ldots,\ y'_{r-1} = \beta y_{r-1} + \beta y_r,\ y_r' = \beta y_r \\ z_1' = \gamma z_1 + \gamma z_2,\ \ldots,\ z'_{r-1} = \gamma z_{r-1} + \gamma z_r,\ z_r' = \gamma z_r \\ w_1' = \delta w_1 + \delta w_2,\ \ldots,\ w'_{r-1} = \delta w_{r-1} + \delta w_r,\ w_r' = \delta w_r \end{array}\right\},$$

unless $\alpha\beta\gamma\delta = 1$.

If $\alpha\beta\gamma\delta = 1$, find the independent quartic invariants in the cases $r = 1, 2$.

[$r = 1$ gives the invariant $x_1 y_1 z_1 w_1$.

$r = 2$ gives $x_2 y_2 z_2 w_2$; $y_2 z_2 (x_1 w_2 - x_2 w_1)$ and two similar invariants; $(y_1 z_2 - y_2 z_1)(x_1 w_2 - x_2 w_1)$ and a similar invariant.]

§ 3. Quadratic Invariants of a Substitution Transformable into its Inverse.

We now consider a substitution in which each invariant-factor $(\lambda - \alpha)^a$, where $\alpha^2 \neq 1$, can be paired with an invariant-factor $(\lambda - \alpha^{-1})^a$. These are of special importance as including orthogonal substitutions.* In fact, if A is orthogonal, A has the same invariant-factors as its inverse; since $A' = A^{-1}$, and A and A' have the same invariant-factors whatever A may be. Therefore, by Ch. II, § 5, Corollary II, A must have its invariant-factors paired as stated.

Suppose then the canonical substitution N has constituents with invariant-factors

$$(\lambda-\alpha)^a,\ (\lambda-\alpha^{-1})^a,\ (\lambda-\alpha)^b,\ (\lambda-\alpha^{-1})^b,\ (\lambda-\alpha)^c,\ (\lambda-\alpha^{-1})^c,\ \ldots$$

where $a \geq b \geq c \geq \ldots$.

The pair of constituents with invariant-factors $(\lambda-\alpha)^a$, $(\lambda-\alpha^{-1})^a$ gives rise to a independent quadratic invariants; the pairs with invariant-factors $(\lambda-\alpha)^a$, $(\lambda-\alpha^{-1})^b$ and $(\lambda-\alpha)^b$, $(\lambda-\alpha^{-1})^a$ to b independent invariants each; the pairs with invariant-factors $(\lambda-\alpha)^a$, $(\lambda-\alpha^{-1})^c$ and $(\lambda-\alpha)^c$, $(\lambda-\alpha^{-1})^a$ to c independent invariants each; ...; the pair with invariant-factors $(\lambda-\alpha)^b$, $(\lambda-\alpha^{-1})^b$ to b independent invariants; the pairs with invariant-factors $(\lambda-\alpha)^b$, $(\lambda-\alpha^{-1})^c$ and $(\lambda-\alpha)^c$ $(\lambda-\alpha^{-1})^b$ to c independent invariants each; and so on.

We thus get

$$(a+2b+2c+2d+\ldots) + (b+2c+2d+\ldots) + (c+2d+\ldots) + \ldots$$

or

$$a+3b+5c+7d+\ldots$$

independent invariants of N.

* But such a substitution is not of necessity transformable into an orthogonal substitution; see Ch. VIII, § 1.

Suppose now that N has constituents with invariant-factors

$$(\lambda-1)^h,\ (\lambda-1)^k,\ (\lambda-1)^l,\ \ldots$$

where

$$h \geq k \geq l \geq \ldots.$$

The constituent with invariant-factor $(\lambda-1)^h$ gives rise to $I\left[\frac{h+1}{2}\right]$ independent invariants; the pair with invariant-factors $(\lambda-1)^h$ and $(\lambda-1)^k$ gives rise to k independent invariants; the pair with invariant-factors $(\lambda-1)^h$ and $(\lambda-1)^l$ gives rise to l independent invariants; and so on.

A similar argument holds for invariant-factors which are powers of $(\lambda+1)$. Hence:—

The number of independent quadratic invariants of a substitution with invariant-factors

$$(\lambda-\alpha)^a,\ (\lambda-\alpha^{-1})^a,\ (\lambda-\alpha)^b,\ (\lambda-\alpha^{-1})^b,\ (\lambda-\alpha)^c,\ (\lambda-\alpha^{-1})^c,\ \ldots,\ \ldots,$$

where

$$a \geq b \geq c \geq \ldots,$$

and with invariant-factors

$$(\lambda-1)^h,\ (\lambda-1)^k,\ (\lambda-1)^l,\ \ldots,$$

where

$$h \geq k \geq l \geq \ldots,$$

and invariant-factors

$$(\lambda+1)^p,\ (\lambda+1)^q,\ (\lambda+1)^r,\ \ldots,$$

where

$$p \geq q \geq r \geq \ldots,$$

is

$$I[\tfrac{1}{2}(h+1)]+I[\tfrac{1}{2}(k+1)]+I[\tfrac{1}{2}(l+1)]+\ldots+(k+2l+3m+\ldots)$$
$$+I[\tfrac{1}{2}(p+1)]+I[\tfrac{1}{2}(q+1)]+I[\tfrac{1}{2}(r+1)]+\ldots+(q+2r+3s+\ldots)$$
$$+\Sigma(a+3b+5c+7d+\ldots),$$

the summation being taken over each distinct pair of reciprocal characteristic-roots of the substitution.

Ex. Find the number of quadratic invariants of the substitutions of Ch. I, § 3, Ex. 6, 7; § 6, Ex. 8; § 9, Ex. 3, 5.

§ 4. Hermitian Invariants.

From the equations of § 1 defining A

$$x_t' = a_{t1}x_1 + a_{t2}x_2 + \ldots + a_{tm}x_m\ (t = 1, 2, \ldots, m) \ \ldots\ldots \text{(i)}$$

we deduce

$$\bar{x}_t' = \bar{a}_{t1}\bar{x}_1 + \bar{a}_{t2}\bar{x}_2 + \ldots + \bar{a}_{tm}\bar{x}_m\ (t = 1, 2, \ldots, m). \ldots\ldots\text{(ii)}$$

If $\Sigma e_{ij}\bar{x}_i x_j$ is an invariant of A, so that

$$\Sigma e_{ij}\bar{x}_i' x_j' \equiv \Sigma e_{ij}\bar{x}_i x_j\ (i, j = 1, 2, \ldots, m),$$

we may consider $\Sigma e_{ij}\bar{x}_i x_j$ as an invariant of the substitution $\mathcal{A}$ of degree $2m$ formed by combining equations (i) and (ii); $x_1, x_2, \ldots, x_m$ and $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ being now considered distinct variables.

If $P^{-1}AP = N$, where N is the canonical form which is the direct product of substitutions of the type

$$x_1' = \alpha x_1 + \alpha x_2,\ \ldots,\ x'_{r-1} = \alpha x_{r-1} + \alpha x_r,\ x_r' = \alpha x_r,$$

and P is

$$x_t' = p_{t1}x_1 + p_{t2}x_2 + \ldots + p_{tm}x_m \quad (t = 1, 2, \ldots, m),$$

then $\mathcal{P}^{-1}\mathcal{A}\mathcal{P} = \mathcal{N}$, where $\mathcal{N}$ is the direct product of

$$\left.\begin{aligned} x_1' &= \alpha x_1 + \alpha x_2,\ \ldots,\ x'_{r-1} = \alpha x_{r-1} + \alpha x_r,\ x_r' = \alpha x_r \\ \bar{x}_1' &= \bar{\alpha}\bar{x}_1 + \bar{\alpha}\bar{x}_2,\ \ldots,\ \bar{x}'_{r-1} = \bar{\alpha}\bar{x}_{r-1} + \bar{\alpha}\bar{x}_r,\ \bar{x}_r' = \bar{\alpha}\bar{x}_r \end{aligned}\right\},$$

and $\mathcal{P}$ is defined by

$$\left.\begin{aligned} x_t' &= p_{t1}x_1 + p_{t2}x_2 + \ldots + p_{tm}x_m \\ \bar{x}_t' &= \bar{p}_{t1}\bar{x}_1 + \bar{p}_{t2}\bar{x}_2 + \ldots + \bar{p}_{tm}\bar{x}_m \end{aligned}\right\} (t = 1, 2, \ldots, m).$$

Hence the invariants $\Sigma e_{ij}\bar{x}_i x_j$ of A (or $\mathcal{A}$) can be deduced from those of N (or $\mathcal{N}$). Such an invariant will be Hermitian when $e_{ij} = \bar{e}_{ji}$.

Now if N is the product of pairs such as (i) of § 2, $\mathcal{N}$ will be the product of substitutions such as

$$\left.\begin{aligned} x_1' &= \alpha x_1 + \alpha x_2,\ \ldots,\ x'_{r-1} = \alpha x_{r-1} + \alpha x_r,\ x_r' = \alpha x_r \\ \bar{y}_1' &= \bar{\beta}\bar{y}_1 + \bar{\beta}\bar{y}_2,\ \ldots,\ \bar{y}'_{s-1} = \bar{\beta}\bar{y}_{s-1} + \bar{\beta}\bar{y}_s,\ \bar{y}_s' = \bar{\beta}\bar{y}_s \end{aligned}\right\}$$

and

$$\left.\begin{aligned} \bar{x}_1' &= \bar{\alpha}\bar{x}_1 + \bar{\alpha}\bar{x}_2,\ \ldots,\ \bar{x}'_{r-1} = \bar{\alpha}\bar{x}_{r-1} + \bar{\alpha}\bar{x}_r,\ \bar{x}_r' = \bar{\alpha}\bar{x}_r \\ y_1' &= \beta y_1 + \beta y_2,\ \ldots,\ y'_{s-1} = \beta y_{s-1} + \beta y_s,\ y_s' = \beta y_s \end{aligned}\right\}.$$

We have then as corresponding invariants of N of the required type when $\alpha\bar{\beta} = 1$ and $r = s$

$$f_1(x, \bar{y}), f_2(x, \bar{y}), \ldots, f_r(x, \bar{y}) \text{ and } f_1(\bar{x}, y), f_2(\bar{x}, y), \ldots, f_r(\bar{x}, y);$$

and similarly when $r \neq s$.

The functions $f_1, f_2, \ldots$ are defined in § 2.

There will be no corresponding invariant when $\alpha\bar{\beta} \neq 1$.

The corresponding *Hermitian* invariants of N are

$$f_r(x, \bar{y}) + f_r(\bar{x}, y), f_{r-1}(x, \bar{y}) + f_{r-1}(\bar{x}, y), f_{r-2}(x, \bar{y}) + f_{r-2}(\bar{x}, y), \ldots,$$

i.e. $\quad \bar{y}_r x_r + y_r \bar{x}_r,\ \bar{y}_r x_{r-1} - \bar{y}_{r-1}x_r + y_r\bar{x}_{r-1} - y_{r-1}\bar{x}_r$, &c.

Take now the case in which N has a constituent

$$x_1' = \alpha x_1 + \alpha x_2.\ \ldots,\ x'_{r-1} = \alpha x_{r-1} + \alpha x_r,\ x_r' = \alpha x_r$$

and therefore $\mathcal{N}$ has the pair of constituents

$$\left.\begin{array}{l} x_1' = \alpha x_1 + \alpha x_2, \ \ldots, \ x'_{r-1} = \alpha x_{r-1} + \alpha x_r, \ x_r' = \alpha x_r \\ \bar{x}_1' = \bar{\alpha}\bar{x}_1 + \bar{\alpha}\bar{x}_2, \ \ldots, \ \bar{x}'_{r-1} = \bar{\alpha}\bar{x}_{r-1} + \bar{\alpha}\bar{x}_r, \ \bar{x}_r' = \bar{\alpha}\bar{x}_r \end{array}\right\}.$$

If $\alpha\bar{\alpha} = 1$,* $\mathcal{N}$ has the invariants

$$f_1(x, \bar{x}), \ f_2(x, \bar{x}), \ \ldots, \ f_r(x, \bar{x}),$$

while there are no such invariants if $\alpha\bar{\alpha} \neq 1$.

The corresponding *Hermitian* invariants of N are

$$f_r(x, \bar{x}), \ i f_{r-1}(x, \bar{x}), \ f_{r-2}(x, \bar{x}), \ i f_{r-3}(x, \bar{x}), \ \ldots,$$

i.e.

$$x_r\bar{x}_r, \ i(x_{r-1}\bar{x}_r - x_r\bar{x}_{r-1}), \ x_{r-2}\bar{x}_r - x_{r-1}\bar{x}_{r-1} + x_r\bar{x}_{r-2} + \tfrac{1}{2}x_{r-1}\bar{x}_r + \tfrac{1}{2}x_r\bar{x}_{r-1}, \ \&c.$$

An argument similar to that of § 3 now shows that:—

For a substitution A on $x_1, x_2, \ldots, x_m$ with invariant-factors

$$(\lambda-\alpha)^a, (\lambda-\bar{\alpha}^{-1})^a, (\lambda-\alpha)^b, (\lambda-\bar{\alpha}^{-1})^b, (\lambda-\alpha)^c, (\lambda-\bar{\alpha}^{-1})^c, \ldots, \ldots,$$

where $a \geq b \geq c \geq \ldots,$

and with invariant-factors

$$(\lambda-\epsilon)^h, \ (\lambda-\epsilon)^k, \ (\lambda-\epsilon)^l, \ \ldots, \ \ldots,$$

where $|\epsilon| = 1$ *and* $h \geq k \geq l \geq \ldots,$

the number of independent invariants of the type $\Sigma e_{ij}\bar{x}_i x_j$ *is*

$$\Sigma\, 2(a+3b+5c+7d+\ldots) + \Sigma(h+3k+5l+\ldots),$$

and the number of Hermitian invariants is

$$\Sigma(a+3b+5c+7d+\ldots) + \Sigma(h+2k+3l+\ldots).$$

The first summation is extended over each pair of characteristic-roots α and $\bar{\alpha}^{-1}$ of A, and the second summation over each characteristic-root ϵ of unit modulus †

Ex. 1. Find the number of Hermitian invariants of the substitutions of Ch. I, § 3, Ex. 5, 6, 7; § 6, Ex. 8; § 9, Ex. 3, 5.

Ex. 2. Any substitution of finite order has a definite Hermitian invariant.

* i.e. $|\alpha| = 1$.

† On the subject of invariants the reader may consult papers by C. Jordan in Liouville's Journal, (1888) p. 349, (1905) p. 217, (1914) p. 97.

CHAPTER VIII

ORTHOGONAL SUBSTITUTIONS

§ 1. Invariant-factors of an Orthogonal Substitution.

WE obtained in Ch. VII, § 2, the most general invariant $\Sigma e_{ij} x_i x_j$ of the canonical substitution N.

If the determinant

$$\begin{vmatrix} e_{11} & e_{12} & . & . & . & e_{1m} \\ e_{21} & e_{22} & . & . & . & e_{2m} \\ . & . & . & . & . & . \\ e_{m1} & e_{m2} & . & . & . & e_{mm} \end{vmatrix} \qquad \text{.....................(i)}$$

of this invariant is not zero, the invariant can be reduced (Ch. III, § 6) by a suitable change of variables to

$$\mathbf{x}_1^2 + \mathbf{x}_2^2 + \ldots + \mathbf{x}_m^2,$$

and the change of variables will convert N into a substitution with $(\mathbf{x}_1^2 + \mathbf{x}_2^2 + \ldots + \mathbf{x}_m^2)$ as invariant, i.e. into an orthogonal substitution.

It will be useful then to determine the conditions under which the determinant (i) does not vanish. From Ch. VII, § 3, it is clear that we may confine ourselves to the case in which N is the direct product of constituents each of which has an invariant-factor which is a power of $\lambda - 1$, or each of which has an invariant-factor which is a power of $\lambda + 1$; and to the case in which N is the direct product of pairs of constituents with invariant-factors such as $(\lambda - \alpha)^a$ and $(\lambda - \alpha^{-1})^a$, $(\alpha^2 \neq 1)$.

(1) Suppose, for instance, that N is

$$x_1' = \alpha(x_1 + x_2),\ x_2' = \alpha(x_2 + x_3),\ x_3' = \alpha x_3;$$
$$\mathbf{x}_1' = \alpha(\mathbf{x}_1 + \mathbf{x}_2),\ \mathbf{x}_2' = \alpha(\mathbf{x}_2 + \mathbf{x}_3),\ \mathbf{x}_3' = \alpha \mathbf{x}_3;$$
$$y_1' = \alpha^{-1}(y_1 + y_2),\ y_2' = \alpha^{-1}(y_2 + y_3),\ y_3' = \alpha^{-1} y_3;$$
$$\mathbf{y}_1' = \alpha^{-1}(\mathbf{y}_1 + \mathbf{y}_2),\ \mathbf{y}_2' = \alpha^{-1}(\mathbf{y}_2 + \mathbf{y}_3),\ \mathbf{y}_3' = \alpha^{-1} \mathbf{y}_3;$$
$$\xi_1' = \alpha(\xi_1 + \xi_2),\ \xi_2' = \alpha \xi_2;\ \boldsymbol{\xi}_1' = \alpha(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2),\ \boldsymbol{\xi}_2' = \alpha \boldsymbol{\xi}_2;$$
$$\Xi_1' = \alpha(\Xi_1 + \Xi_2),\ \Xi_2' = \alpha \Xi_2;$$
$$\eta_1' = \alpha^{-1}(\eta_1 + \eta_2),\ \eta_2' = \alpha^{-1}\eta_2;\ \boldsymbol{\eta}_1' = \alpha^{-1}(\boldsymbol{\eta}_1 + \boldsymbol{\eta}_2),\ \boldsymbol{\eta}_2' = \alpha^{-1}\boldsymbol{\eta}_2;$$
$$\mathrm{H}_1' = \alpha^{-1}(\mathrm{H}_1 + \mathrm{H}_2),\ \mathrm{H}_2' = \alpha^{-1}\mathrm{H}_2.$$

Let any quadratic invariant be (with the notation of Ch. VII, § 2)

$$\begin{aligned}\{&af_1(x, y)+bf_1(\mathbf{x}, y)+c(x, \mathbf{y})+df_1(\mathbf{x}, \mathbf{y})\\ &+lf_1(\xi, \eta)+mf_1(\boldsymbol{\xi}, \eta)+nf_1(\Xi, \eta)+pf_1(\xi, \boldsymbol{\eta})+qf_1(\boldsymbol{\xi}, \boldsymbol{\eta})\\ &\qquad+rf_1(\Xi, \boldsymbol{\eta})+uf_1(\xi, \mathrm{H})+vf_1(\boldsymbol{\xi}, \mathrm{H})+wf_1(\Xi, \mathrm{H}\}\\ &\qquad\qquad+\{a'f_2(x, y)+\ldots\}+\{a''f_3(x, y)+\ldots\}.\end{aligned}$$

The determinant of this invariant is symmetric and of order 24, the first 12 elements in the first 12 rows and the last 12 elements in the last 12 rows being zero.* Hence the determinant is minus the square of the minor determinant formed by the last 12 elements of the first 12 rows (or the first 12 elements of the last 12 rows), which is of the type

$$\begin{vmatrix}
\circ & \circ & a & \circ & \circ & b & \circ & \circ & \circ & \circ & \circ & \circ\\
\circ & -a & \ast & \circ & -b & \ast & \circ & \ast & \circ & \ast & \circ & \ast\\
a & \ast & \ast & b & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast\\
\circ & \circ & c & \circ & \circ & d & \circ & \circ & \circ & \circ & \circ & \circ\\
\circ & -c & \ast & \circ & -d & \ast & \circ & \ast & \circ & \ast & \circ & \ast\\
c & \ast & \ast & d & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast\\
\circ & \circ & \ast & \circ & \circ & \ast & \circ & -l & \circ & -m & \circ & -n\\
\circ & \ast & \ast & \circ & \ast & \ast & l & \ast & m & \ast & n & \ast\\
\circ & \circ & \ast & \circ & \circ & \ast & \circ & -p & \circ & -q & \circ & -r\\
\circ & \ast & \ast & \circ & \ast & \ast & p & \ast & q & \ast & r & \ast\\
\circ & \circ & \ast & \circ & \circ & \ast & \circ & -u & \circ & -v & \circ & -w\\
\circ & \ast & \ast & \circ & \ast & \ast & u & \ast & v & \ast & w & \ast
\end{vmatrix} \quad \ldots \text{(ii)}$$

the asterisks denoting certain quantities which are not zero in general.

Now rearrange rows and columns in (ii) so that the rows and columns now in the order 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 take respectively the orders 5, 3, 1, 6, 4, 2, 10, 7, 11, 8, 12, 9 and 1, 3, 5, 2, 4, 6, 7, 10, 8, 11, 9, 12.

The determinant (ii) now becomes

$$\left.\begin{matrix}
a & b & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast\\
c & d & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast\\
\circ & \circ & -a & -b & \ast & \ast & \circ & \circ & \circ & \ast & \ast & \ast\\
\circ & \circ & -c & -d & \ast & \ast & \circ & \circ & \circ & \ast & \ast & \ast\\
\circ & \circ & \circ & \circ & a & b & \circ & \circ & \circ & \circ & \circ & \circ\\
\circ & \circ & \circ & \circ & c & d & \circ & \circ & \circ & \circ & \circ & \circ\\
\circ & \circ & \ast & \ast & \ast & \ast & l & m & n & \ast & \ast & \ast\\
\circ & \circ & \ast & \ast & \ast & \ast & p & q & r & \ast & \ast & \ast\\
\circ & \circ & \ast & \ast & \ast & \ast & u & v & w & \ast & \ast & \ast\\
\circ & \circ & \circ & \circ & \ast & \ast & \circ & \circ & \circ & -l & -m & -n\\
\circ & \circ & \circ & \circ & \ast & \ast & \circ & \circ & \circ & -p & -q & -r\\
\circ & \circ & \circ & \circ & \ast & \ast & \circ & \circ & \circ & -u & -v & -w
\end{matrix}\right|$$

* For there is no invariant such as $f_1(x, x)$, &c., since $\alpha^2 \neq 1$.

$$= -\begin{vmatrix} a & b \\ c & d \end{vmatrix}^3 \times \begin{vmatrix} l & m & n \\ p & q & r \\ u & v & w \end{vmatrix}^{2} *.$$

Hence neither

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ nor } \begin{vmatrix} l & m & n \\ p & q & r \\ u & v & w \end{vmatrix}$$

must vanish; and a similar result holds in the general case. The reader will easily verify in a similar manner that (ii) always vanishes unless the invariant-factors of N are grouped as in Ch. VII, § 3. This agrees with the theorem proved in that section for orthogonal substitutions.

(2) Now suppose that N is, for instance,

$$x_1' = x_1 + x_2,\ x_2' = x_2 + x_3,\ x_3' = x_3;$$

$$\mathbf{x}_1' = \mathbf{x}_1 + \mathbf{x}_2,\ \mathbf{x}_2' = \mathbf{x}_2 + \mathbf{x}_3,\ \mathbf{x}_3' = \mathbf{x}_3;$$

$$\xi_1' = \xi_1 + \xi_2,\ \xi_2' = \xi_2;\ \boldsymbol{\xi}_1' = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2,\ \boldsymbol{\xi}_2' = \boldsymbol{\xi}_2;$$

$$\Xi_1' = \Xi_1 + \Xi_2,\ \Xi_2' = \Xi_2.$$

The most general quadratic invariant has now a determinant such as (ii), but with

$$b = c,\ l = q = w = 0,† \ p = -m,\ u = -n,\ v = -r;$$

so that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ is symmetric and } \begin{vmatrix} l & m & n \\ p & q & r \\ u & v & w \end{vmatrix}$$

skew-symmetric.

But a skew-symmetric determinant of odd order vanishes.‡ Hence an orthogonal substitution cannot have an *odd* number of invariant-factors of the type $(\lambda-1)^r$ or an *odd* number of the type $(\lambda+1)^r$, where r is a given even integer.

Hence:—

A substitution is transformable into an orthogonal substitution if and only if it has not got exactly $(2p+1)$ invariant-factors of the type $(\lambda-1)^r$ or of the type $(\lambda+1)^r$, where r is a given even *integer, while its invariant-factors which are not powers of $(\lambda-1)$ or $(\lambda+1)$ occur in pairs of the type $(\lambda-\alpha)^a$, $(\lambda-\alpha^{-1})^a$.*

It is easy to construct an orthogonal substitution with given invariant-factors satisfying the conditions expressed in this

* Cf. Ch. VI, § 4, Corollary II.
† Since $f_1(\xi, \xi)$, $f_1(\boldsymbol{\xi}, \boldsymbol{\xi})$, $f_1(\Xi, \Xi)$ vanish identically.
‡ Otherwise changing rows into columns would alter its sign.

theorem. We have only to write down a canonical substitution N with the given invariant-factors, take any invariant with non-zero determinant, and put this invariant in the form $\xi_1^2+\xi_2^2+\xi_3^2+\dots$ by Ch. III, § 6.

Now express N in terms of $\xi_1, \xi_2, \xi_3, \dots$, and it becomes the required orthogonal substitution.

Ex. 1. Find an orthogonal substitution with invariant-factors $(\lambda-\alpha)$, $(\lambda-\alpha^{-1})$.

[$x'=\alpha x$, $y'=\alpha^{-1}y$ has the invariant

$$xy \equiv \left(\frac{x+y}{2}\right)^2+\left(\frac{ix-iy}{2}\right)^2.$$

Hence if we express $x'=\alpha x$, $y'=\alpha^{-1}y$ in terms of

$$\xi=\tfrac{1}{2}(x+y),\quad \eta=\tfrac{1}{2}(ix-iy),$$

we get the required orthogonal substitution. It is

$$\xi'=\frac{\alpha+\alpha^{-1}}{2}\xi-i\frac{\alpha-\alpha^{-1}}{2}\eta,\quad \eta'=i\frac{\alpha-\alpha^{-1}}{2}\xi+\frac{\alpha+\alpha^{-1}}{2}\eta.]$$

Ex. 2. Find an orthogonal substitution with the invariant-factor $(\lambda-1)^3$, and an orthogonal substitution with the invariant-factors $(\lambda-1)^2$, $(\lambda-1)^2$.

§ 2. Transformation of one Orthogonal Substitution into another.

Just as in Ch. VI, §§ 5 and 6, we proved that it was possible to transform the symmetric substitution A by a suitable change of variables so that A became a canonical substitution and $x_1^2+x_2^2+\dots+x_m^2$ took a certain canonical shape, so we can prove that:—

If A is orthogonal, *we can transform A into the canonical substitution N, which is the direct product of substitutions of the type*

$$\left.\begin{aligned} x_1' &= \alpha(x_1+x_2), \quad \dots, x'_{r-1}=\alpha(x_{r-1}+x_r), \quad x_r'=\alpha x_r \\ y_1' &= \alpha^{-1}(y_1+y_2), \dots, y'_{r-1}=\alpha^{-1}(y_{r-1}+y_r), \ y_r'=\alpha^{-1}y_r \end{aligned}\right\},$$

where either $\alpha^2\neq 1$ or else $\alpha^2=1$ and r is even, and of substitutions of the type

$$X_1'=\alpha(X_1+X_2), \dots, X'_{r-1}=\alpha(X_{r-1}+X_r), X_r'=\alpha X_r,$$

where $\alpha^2=1$ and r is odd; and at the same time transform $x_1^2+x_2^2+\dots+x_m^2$ into the sum σ of functions of the type

$f_1(x, y)$ and $f_1(X, X)$ respectively; f_1 being defined as in Ch. VII, § 2.

The theorem will be proved in § 3.

We shall assume its truth for the present and deduce the theorem :—

Any given orthogonal substitution can be transformed into another given orthogonal substitution with the same invariant-factors by an orthogonal substitution.

For we can choose substitutions P, Q such that

$$P^{-1}NP = A,\ Q^{-1}NQ = B,$$

where N is the canonical form of A or B, and such that P and Q transform $x_1^2 + x_2^2 + \ldots + x_m^2$ into σ.

If C is the substitution corresponding to the bilinear form σ, we have, by Ch. III, § 1,

$$PP' = C,\ QQ' = C.$$

Hence, as in Ch. VI, § 5, the orthogonal substitution $P^{-1}Q$ transforms A into B.

Again :—

Any given quadratic invariant with non-zero determinant of a substitution A can be transformed into any other given quadratic invariant with non-zero determinant by a change of variables which does not alter the substitution.

Let the substitutions corresponding to the two given invariants h and k of A be H and K, and let P and Q be the substitutions transforming h and k into σ and such that

$$P^{-1}NP = A,\ Q^{-1}NQ = A.$$

Then, by Ch. III, § 1, we have

$$PHP' = C,\ QKQ' = C.$$

Hence $Q^{-1}P$ transforms h into k and is permutable with A, for

$$(Q^{-1}P)H(Q^{-1}P)' = Q^{-1}.PHP'.Q'^{-1} = Q^{-1}CQ'^{-1} = K,$$

and

$$(Q^{-1}P)^{-1}A(Q^{-1}P) = P^{-1}.QAQ^{-1}.P = P^{-1}NP = A.$$

Ex. 1. An orthogonal substitution can be transformed by an orthogonal substitution into the direct product of an orthogonal substitution with no linear invariant-factor and of substitutions of the three types

(i) $x' = x$, (ii) $x' = -x$,

(iii) $x' = \cos\theta.x - \sin\theta.y,\ y' = \sin\theta.x + \cos\theta.y.$

[It is evident that an orthogonal substitution can be constructed of this type having the same invariant-factors as the given substitution.]

Ex. 2. A given real orthogonal substitution can be transformed into another given real orthogonal substitution with the same invariant-factors by a real orthogonal substitution.

[Use Ch. I, § 15.]

Ex. 3. Transform the real orthogonal substitution

$$9x' = 4x-y+8z,\ 9y' = -7x+4y+4z,\ 9z' = -4x-8y+z$$

into $$x' = -y,\ y' = x,\ z' = z.$$

[Transform by

$$3x' = x+2y+2z,\ 3y' = 2x+y-2z,\ 3z' = 2x-2y+z.]$$

Ex. 4. Transform

$$2x^2+7y^2+20z^2+24yz+12zx+7xy$$

into $$4x^2+15y^2+52z^2+56yz+28zx+15xy$$

without altering the substitution

$$A \equiv (3x+4y+8z,\ -3x-3y-8z,\ x+y+3z),$$

of which they are invariants.

[If $P \equiv (-2y-z,\ 2x+3y+2z,\ -x-y-z)$,

$Q \equiv (2y-z,\ -2x-y-3z,\ x+2z)$,

$N \equiv (x+y,\ y+z,\ z),\quad C \equiv (-z,\ y-\frac{1}{2}z,\ -x-\frac{1}{2}y)$,

$H \equiv (2x+\frac{7}{2}y+6z,\ \frac{7}{2}x+7y+12z,\ 6x+12y+20z)$,

$K \equiv (4x+\frac{15}{2}y+14z,\ \frac{15}{2}x+15y+28z,\ 14x+28y+52z)$,

we have $P^{-1}NP = A,\ Q^{-1}NQ = A,\ PHP' = C,\ QHQ' = C.$

Hence the required transforming substitution permutable with A is $$Q^{-1}P \equiv (-3x-4y-8z,\ x-y.\ y+z).]$$

§ 3.

We now prove the first theorem of § 2. The proof is very similar to that of Ch. VI, §§ 6 and 10, except that A is orthogonal instead of symmetric or alternate.

Suppose that, when we replace

$$x_t \text{ by } r_{t1}x_1 + r_{t2}x_2 + \dots + r_{tm}x_m \quad (t = 1, 2, \dots, m),$$

A becomes N and $x_1^2+x_2^2+\dots+x_m^2$ becomes $\Sigma f_{ij}x_ix_j$.

Then $R^{-1}NR = A$ and $RR' = F$ (Ch. I, § 5, and Ch. III, § 1).

By Ch. VII, § 1, $\Sigma f_{ij}x_ix_j$ is an invariant of N, since $x_1^2+x_2^2+\dots+x_m^2$ is an invariant of A.

Assume now that D can be chosen so that, when we replace

$$x_t \text{ by } d_{t1}x_1 + d_{t2}x_2 + \ldots + d_{tm}x_m \ (t = 1, 2, \ldots, m),$$

N is unaltered and $\Sigma f_{ij}x_ix_j$ becomes σ.*

Then $D^{-1}ND = N$ and $DFD' = C$.

Hence, if

$$P = DR,\ P^{-1}NP = R^{-1}\,.\,D^{-1}ND\,.\,R = R^{-1}NR = A,$$

and

$$PP' = DR\,.\,R'D' = DFD' = C,$$

so that, when we replace

$$x_t \text{ by } p_{t1}x_1 + p_{t2}x_2 + \ldots + p_{tm}x_m \ (t = 1, 2, \ldots, m),$$

A becomes N and $x_1^2 + x_2^2 + \ldots + x_m^2$ becomes σ.

It remains then to justify the choice of D. As in Ch. VI, § 6, we may confine ourselves to the case in which N has only one distinct characteristic-root.

If we perform the transformation (iv) of Ch. VI, § 6, on $f_1(x, y)$, we obtain

$$\{f_r(a, b)\,.\,f_1(x, y) + f_{r-1}(a, b)\,.\,f_2(x, y) + f_{r-2}(a, b)\,.\,f_3(x, y) + \ldots + f_1(a, b)\,.\,f_r(x, y)\}$$

when r is even, and

$$\{f_r(a, b)\,.\,f_1(x, y) + f_{r-1}(a, b)\,.\,f_2(x, y) + f_{r-2}(a, b)\,.\,f_3(x, y) + \ldots + f_1(a, b)\,.\,f_r(x, y)\}$$
$$+ \tfrac{1}{4}\{f_{r-1}(a, b) f_4(x, y) + f_{r-3}(a, b)\,.\,f_6(x, y) + \ldots + f_4(a, b)\,.\,f_{r-1}(x, y)\}$$

when r is odd.

This may be proved by induction as follows:—

The result is easily verified to be true when $r = 1, 2, 3, 4, 5$. Assume it true for all values of r up to the one considered.

Since $f_1(x, y)$ is an invariant of

$$\left.\begin{array}{l} x_1' = \alpha(x_1 + x_2),\ \ldots,\ x'_{r-1} = \alpha(x_{r-1} + x_r),\ x_r' = \alpha x_r \\ y_1' = \beta(y_1 + y_2),\ \ldots,\ y'_{r-1} = \beta(y_{r-1} + y_r),\ y_r' = \beta y_r \end{array}\right\}, \ldots \text{(i)}$$

where $\alpha\beta = 1$, and since the substitution (iv) of Ch. VI, § 6, is permutable with N, therefore $f_1(x, y)$ must be transformed into another invariant of N, e.g.

$$k_1 f_1(x, y) + k_2 f_2(x, y) + k_3 f_3(x, y) + \ldots + k_{r-1} f_{r-1}(x, y) + k_r f_r(x, y).$$

If in $f_t(x, y)$ we put x_r and y_r zero, we get the result of changing $x_1, x_2, x_3, \ldots, x_r$ into $0, x_1, x_2, \ldots, x_{r-1}$ and $y_1, y_2, y_3, \ldots, y_r$ into $0, y_1, y_2, \ldots, y_{r-1}$ in $-f_{t+2}(x, y)$, as is obvious from the matrix of $f_t(x, y)$ as given in Ch. VII, § 2.

* Whatever substitution permutable with N D may be, D transforms $\Sigma f_{ij}x_ix_j$ into an invariant of N.

Hence by the given transformation the invariant $f_1(x, y)$ of

$$\left.\begin{aligned} x_1' &= \alpha(x_1+x_2), \ldots, x'_{r-2} = \alpha(x_{r-2}+x_{r-1}), x'_{r-1} = \alpha x_{r-1} \\ y_1' &= \beta(y_1+y_2), \ldots, y'_{r-2} = \beta(y_{r-2}+y_{r-1}), y'_{r-1} = \beta y_{r-1} \end{aligned}\right\}$$

is transformed into

$$k_1 f_1(x, y) + k_2 f_2(x, y) + \ldots + k_{r-2} f_{r-2}(x, y).$$

But the result is assumed true when r has any value less than the one considered, so that $k_1, k_2, \ldots, k_{r-2}$ have the values stated in the theorem we wish to prove.

Also, when we actually perform the transformation on $f_1(x, y)$, we can pick out the coefficients of $f_{r-1}(x, y)$ and $f_r(x, y)$, thus obtaining $k_r = f_1(a, b)$ and $k_{r-1} = f_2(a, b)$ or $f_2(a, b) + \frac{1}{4} f_4(a, b)$ as r is even or odd.

(1) Suppose we first consider a case such as that in which N is

$$x_1' = \alpha(x_1+x_2),\ x_2' = \alpha(x_2+x_3),\ x_3' = \alpha x_3;$$
$$y_1' = \beta(y_1+y_2),\ y_2' = \beta(y_2+y_3),\ y_3' = \beta y_3;$$
$$\left.\begin{aligned} \xi_1' &= \alpha(\xi_1+\xi_2),\ \xi_2' = \alpha\xi_2 \\ \eta_1' &= \beta(\eta_1+\eta_2),\ \eta_2' = \beta\eta_2 \end{aligned}\right\},$$

where $\alpha\beta = 1$; so that N is the direct product of a pair of substitutions of degree 3 and a pair of lower degree 2.

Any invariant of N takes the form

$$\{af_1(x, y) + a'f_2(x, y) + a''f_3(x, y)\} + \{df_1(\xi, \eta) + d'f_2(\xi, \eta)\}$$
$$+ \{bf_1(x, \eta) + b'f_2(x, \eta)\} + \{cf_1(\xi, y) + c'f_2(\xi, y)\}.$$

Now put

$$\left.\begin{aligned} \xi_1 + kx_2 + k'x_3 \text{ for } \xi_1,\ \xi_2 + kx_3 \text{ for } \xi_2 \\ \eta_1 + ly_2 + l'y_3 \text{ for } \eta_1,\ \eta_2 + ly_3 \text{ for } \eta_3 \end{aligned}\right\}.$$

This will not alter N, and will transform the invariant into

$$\{Af_1(x, y) + \ldots + \ldots\} + \{df_1(\xi, \eta) + \ldots\} + \{Bf_1(x, \eta) + B'f_2(x, \eta)\}$$
$$+ \{Cf_1(\xi, y) + C'f_2(\xi, y)\}.$$

Then we can choose k, k', l, l' so that B, B', C, C' vanish; and the invariant is the sum of a bilinear form on x, y, and a bilinear form on ξ, η.

Similarly in such a case as that in which N is

$$\left.\begin{aligned} x_1' &= \alpha(x_1+x_2),\ x_2' = \alpha(x_2+x_3),\ x_3' = \alpha x_3; \\ \xi_1' &= \alpha(\xi_1+\xi_2),\ \xi_2' = \alpha\xi_2;\ X_1' = \alpha(X_1+X_2),\ X_2' = \alpha X_2 \end{aligned}\right\},$$

where $\alpha^2 = 1$, we put

$$\left.\begin{aligned} \xi_1 + kx_2 + k'x_3 \text{ for } \xi_1,\ \xi_2 + kx_3 \text{ for } \xi_2 \\ X_1 + lx_2 + l'x_3 \text{ for } X_1,\ X_2 + lx_3 \text{ for } X_2 \end{aligned}\right\},$$

and then choose k, k', l, l' so that the invariant becomes the sum of a quadratic form on x_1, x_2, x_3, and a quadratic form on ξ_1, ξ_2, X_1, X_2.

(2) We can now confine ourselves to the case in which N is the direct product of constituents each of which is of the same degree.

Suppose N is, for instance,

$$\left.\begin{array}{l} x_1' = \alpha(x_1+x_2),\ \ldots,\ x'_{r-1} = \alpha(x_{r-1}+x_r),\ x_r' = \alpha x_r \\ y_1' = \beta(y_1+y_2),\ \ldots,\ y'_{r-1} = \beta(y_{r-1}+y_r),\ y_r' = \beta y_r \\ \xi_1' = \alpha(\xi_1+\xi_2),\ \ldots,\ \xi'_{r-1} = \alpha(\xi_{r-1}+\xi_r),\ \xi_r' = \alpha\xi_r \\ \eta_1' = \beta(\eta_1+\eta_2),\ \ldots,\ \eta'_{r-1} = \beta(\eta_{r-1}+\eta_r),\ \eta_r' = \beta\eta_r \end{array}\right\},$$

where $\alpha\beta = 1$.

Any invariant of non-zero determinant is

$$\{af_1(x, y) + a'f_2(x, y) + a''f_3(x, y) + \ldots\} + \{bf_1(x, \eta) + \ldots\} \\ + \{cf_1(\xi, y) + \ldots\} + \{df_1(\xi, \eta) + \ldots\},$$

where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0.$

Put

$$\left.\begin{array}{l} kx_1 + l\xi_1 \text{ for } x_1,\ kx_2 + l\xi_2 \text{ for } x_2,\ \ldots,\ kx_r + l\xi_r \text{ for } x_r \\ mx_1 + n\xi_1 \text{ for } \xi_1,\ mx_2 + n\xi_2 \text{ for } x_2,\ \ldots,\ mx_r + n\xi_r \text{ for } \xi_r \end{array}\right\}.$$

Then N is unaltered and the invariant becomes

$$\{Af_1(x, y) + A'f_2(x, y) + A''f_3(x, y) + \ldots\} + \{Bf_1(x, \eta) + \ldots\} \\ + \{Cf_1(\xi, y) + \ldots\} + \{Df_1(\xi, \eta) + \ldots\},$$

where

$$A = ak + cm,\ B = bk + dm,\ C = al + cn,\ D = bl + dn.$$

We can choose k, l, m, n to make $A = D = 1$, $B = C = 0$; and then we can apply the method of (1) to transform the invariant into the sum of a bilinear form on x, y and a bilinear form on ξ, η without altering N.

Suppose, again, N is

$$\left.\begin{array}{l} x_1' = \alpha(x_1+x_2),\ x_2' = \alpha x_2;\ x_3' = \alpha(x_3+x_4),\ x_4' = \alpha x_4 \\ x_5' = \alpha(x_5+x_8),\ x_6' = \alpha x_8;\ x_7' = \alpha(x_7+x_8),\ x_8' = \alpha x_8 \end{array}\right\},$$

where $\alpha^2 = 1$, being the product of four constituents each of even order 2.

The determinant of any invariant of N is of the type

$$\begin{vmatrix} 0 & r_{11} & 0 & r_{12} & 0 & r_{13} & 0 & r_{14} \\ -r_{11} & * & -r_{12} & * & -r_{13} & * & -r_{14} & * \\ 0 & r_{21} & 0 & r_{22} & 0 & r_{23} & 0 & r_{24} \\ -r_{21} & * & -r_{22} & * & -r_{23} & * & -r_{24} & * \\ 0 & r_{31} & 0 & r_{32} & 0 & r_{33} & 0 & r_{34} \\ -r_{31} & * & -r_{32} & * & -r_{33} & * & -r_{34} & * \\ 0 & r_{41} & 0 & r_{42} & 0 & r_{43} & 0 & r_{44} \\ -r_{41} & * & -r_{42} & * & -r_{43} & * & -r_{44} & * \end{vmatrix}$$

(the asterisks denoting as before quantities not necessarily zero), where

$$\begin{vmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{vmatrix}$$

is skew-symmetric.

Put now

$$\begin{aligned} &a_{11}x_1 + a_{12}x_3 + a_{13}x_5 + a_{14}x_7 \text{ for } x_1, \\ &a_{21}x_1 + a_{22}x_3 + a_{23}x_5 + a_{24}x_7 \text{ for } x_3, \\ &a_{31}x_1 + a_{32}x_3 + a_{33}x_5 + a_{34}x_7 \text{ for } x_5, \\ &a_{41}x_1 + a_{42}x_3 + a_{43}x_5 + a_{44}x_7 \text{ for } x_7, \end{aligned}$$

$$\left.\begin{aligned} &a_{11}x_2 + a_{12}x_4 + a_{13}x_6 + a_{14}x_8 \text{ for } x_2 \\ &a_{21}x_2 + a_{22}x_4 + a_{23}x_6 + a_{24}x_8 \text{ for } x_4 \\ &a_{31}x_2 + a_{32}x_4 + a_{33}x_6 + a_{34}x_8 \text{ for } x_6 \\ &a_{41}x_2 + a_{42}x_4 + a_{43}x_6 + a_{44}x_8 \text{ for } x_8 \end{aligned}\right\},$$

the constants a_{ij} being chosen so that the substitution of

$$\left.\begin{aligned} &a_{t1}x_1 + a_{t2}x_2 + a_{t3}x_3 + a_{t4}x_4 \text{ for } x_t \\ &a_{t1}y_1 + a_{t2}y_2 + a_{t3}y_3 + a_{t4}y_4 \text{ for } y_t \end{aligned}\right\} \ (t = 1, 2, 3, 4)$$

reduces the alternate bilinear form $\Sigma\, r_{ij} y_i x_j$ to

$$\begin{aligned} &\Sigma\, r_{ij}\,(a_{i1}y_1 + a_{i2}y_2 + a_{i3}y_3 + a_{i4}y_4)\,(a_{j1}x_1 + a_{j2}x_2 + a_{j3}x_3 + a_{j4}x_4) \\ &\qquad \equiv (y_1x_2 - y_2x_1) + (y_3x_4 - y_4x_3) + (y_5x_6 - y_6x_5) + (y_7x_8 - y_8x_7).* \end{aligned}$$

Then the matrix of the invariant takes the form

$$\begin{vmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & * & -1 & * & 0 & * & 0 & * \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & * & 0 & * & 0 & * & -1 & * \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & * & 0 & * & 1 & * & 0 & * \end{vmatrix}.$$

* See Ch. III, § 7.

We now put

$$\left.\begin{array}{l} kx_1+k'x_3+x_5 \text{ for } x_5,\ kx_2+k'x_4+x_6 \text{ for } x_6 \\ lx_1+l'x_3+x_7 \text{ for } x_7,\ lx_2+l'x_4+x_8 \text{ for } x_8 \end{array}\right\},$$

and by a proper choice of k, k', l, l' we can reduce the invariant to the sum of a quadratic form on x_1, x_2, x_3, x_4, and a quadratic form on x_5, x_6, x_7, x_8 as in (1).

If N is the product of constituents each of the same *odd* order, the procedure is nearly identical with that used when discussing symmetric substitutions in Ch. VI, § 6, (2).

(3) We can now confine ourselves to the case in which N has a single pair of invariant-factors $(\lambda-\alpha)^r$, $(\lambda-\alpha^{-1})^r$ where $\alpha^2 \neq 1$, a pair of invariant-factors $(\lambda-\alpha)^r$, $(\lambda-\alpha)^r$ where $\alpha^2=1$ and r is even, or a single invariant-factor $(\lambda-\alpha)^r$ where $\alpha^2=1$ and r is odd.

We will take one of these cases, leaving the other two as an exercise to the reader.

Suppose, for instance, N is

$$x_1'=\alpha(x_1+x_2),\ \ldots,\ x'_{r-1}=\alpha(x_{r-1}+x_r),\ x_r'=\alpha x_r,\ \ldots \quad \text{(ii)}$$

where $\alpha^2=1$ and r is odd.

Any invariant of N of non-zero determinant is

$$k_1 f_1(x,x)+k_3 f_3(x,x)+\ldots+k_r f_r(x,x), \text{ where } k_1 \neq 0,$$

remembering that, if r is odd, $f_2(x,x)$, $f_4(x,x)$, $f_6(x,x)$, ..., vanish identically.

Put

$$\left.\begin{array}{r} a_r x_1+a_{r-1}x_2+a_{r-2}x_3+\ldots+a_1x_r \text{ for } x_1 \\ a_r x_2 \quad +a_{r-1}x_3+\ldots+a_2x_r \text{ for } x_2 \\ \cdots\cdots\cdots\cdots \\ a_r x_r \text{ for } x_r \end{array}\right\}.$$

This does not alter N and reduces the invariant to

$$K_1 f_1(x,x)+K_3 f_3(x,x)+\ldots+K_r f_r(x,x)$$

where $K_1=k_1 f_r(a,a)$, $K_2=k_1 f_{r-2}(a,a)+k_3 f_r(a,a)$,

$$K_3=k_1 f_{r-4}(a,a)+k_3 f_{r-2}(a,a)+k_5 f_r(a,a), \ldots \quad \ldots\ldots\text{(iii)}$$

Choose $\qquad a_{r-1}=a_{r-3}=a_{r-5}=\ldots=0$

(or take any other arbitrarily chosen values). Then we can choose in turn $a_r, a_{r-2}, a_{r-4}, \ldots$ to satisfy the equations (iii) when we take

$$K_1=1,\ K_2=K_3=K_4=\ldots=0,$$

remembering that

$$f_r(a,a)=a_r^2,\ f_{r-2}(a,a)=2a_r a_{r-2}-a^2_{r-1}+a_{r-1}a_r, \text{ \&c.}$$

We have thus reduced any invariant of N to the form $f_1(x, x)$ without altering N, as required.

Ex. 1. Find the number of orthogonal substitutions permutable with a given orthogonal substitution having a single invariant-factor $(\lambda-1)^r$, where r is odd.

[It is sufficient to find all substitutions permutable with

$$x_1' = x_1 + x_2, \ \ldots, \ x'_{r-1} = x_{r-1} + x_r, \ x_r' = x_r,$$

and having $f_1(x, x)$ as an invariant.]

Ex. 2. Find the number, if the given orthogonal substitution has the pair of invariant-factors $(\lambda-\alpha)^r$, $(\lambda-\alpha^{-1})^r$.

[Other cases may be taken as examples; for instance, the case in which the given substitution has the pair of invariant-factors $(\lambda-1)^r$, $(\lambda-1)^r$, where r is even.]

Ex. 3. If we perform the transformation (iv) of Ch. VI, § 6, on the invariant $\psi_1(x, y)$ of Ch. VII, § 2, Ex. 1, we get

$$\psi_r(a, b) \, . \, \psi_1(x, y) + \psi_{r-1}(a, b) \, . \, \psi_2(x, y) + \ldots + \psi_1(a, b) \, . \, \psi_r(x, y).$$

[Put $x_r = 0$, and use induction.]

Ex. 4. Show that the substitution (i) of § 3 is transformable into its inverse by the substitution K

$$\left.\begin{array}{l} x_1' = -y_1 + y_2 - y_3 + y_4 - y_5 + \ldots \\ x_2' = y_2 - {}^2C_1 y_3 + {}^3C_1 y_4 - {}^4C_1 y_5 + \ldots \\ x_3' = -y_3 + {}^3C_2 y_4 - {}^4C_2 y_5 + \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_1' = -x_1 + x_2 - x_3 + x_4 - x_5 + \ldots \\ y_2' = x_2 - {}^2C_1 x_3 + {}^3C_1 x_4 - {}^4C_1 x_5 + \ldots \\ y_3' = -x_3 + {}^3C_2 x_4 - {}^4C_2 x_5 + \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}\right\}.$$

Show that $K^2 = E$, and that K has $f_1(x, y)$ as an invariant.

Prove a similar result for the substitution (ii) of § 3. (See Ch. I, § 5, Ex. 5).

Deduce the theorem of Ch. II, § 7; and show that if in that section A is *orthogonal*, A is the product of two *orthogonal* substitutions of order 2 (which are therefore also symmetric), each of which transforms A into its inverse.

§ 4. Substitutions with $\mathbf{x_1 \bar{x}_1 + \ldots + x_k \bar{x}_k - x_{k+1} \bar{x}_{k+1} - \ldots - x_m \bar{x}_m}$ as Invariant.

As in § 1, we prove that:—

A substitution is transformable into a substitution with a Hermitian invariant of non-zero determinant if and only if its invariant-factors, which are not of the type $(\lambda-\epsilon)^r$, *where* $|\epsilon| = 1$, *occur in pairs of the type* $(\lambda-\alpha)^a$, $(\lambda-\bar{\alpha}^{-1})^a$.

If the canonical form N of such a substitution A is the direct product of substitutions of the type

$$\left.\begin{array}{l} x_1' = \alpha(x_1+x_2), \quad \ldots, x'_{r-1} = \alpha(x_{r-1}+x_r), \quad x_r' = \alpha x_r \\ y_1' = \bar{\alpha}^{-1}(y_1+y_2), \ldots, y'_{r-1} = \bar{\alpha}^{-1}(y_{r-1}+y_r), y_r' = \bar{\alpha}^{-1} y_r \end{array}\right\},$$

where $|\alpha| \neq 1$, and of substitutions of the type

$$X_1' = \alpha(X_1+X_2), \ldots, X'_{r-1} = \alpha(X_{r-1}+X_r), X_r' = \alpha X_r,$$

where $|\alpha| = 1$, we can, without altering N, transform any given Hermitian invariant of N with non-zero determinant into the sum σ of functions of the type

$$f_1(x, \bar{y}) + f_1(\bar{x}, y) \quad \text{and} \quad \pm(i)^{r-1} f_1(X, \bar{X})$$

respectively.

The proof is similar to that of §§ 2, 3, but is somewhat easier, as we escape the complication introduced by the fact that $f_1(x, x) \equiv 0$, if r is even.

As before, it will follow that A can be transformed into N while a given Hermitian invariant of A with non-zero determinant is transformed into σ.

Now either $f_1(x, \bar{y}) + f_1(\bar{x}, y)$ or $\pm(i)^{r-1} f_1(X, \bar{X})$, when put in standard form by the method of Ch. III, § 6, is of the type

$$\pm(\eta_1\bar{\eta}_1 - \eta_2\bar{\eta}_2 + \eta_3\bar{\eta}_3 - \eta_4\bar{\eta}_4 + \ldots).$$

Take, for instance, the case of $r = 5$.

$$\begin{aligned} (i)^{r-1} f_1(X, \bar{X}) \equiv\; & \bar{X}_1 X_5 + \bar{X}_2(-X_4 + \tfrac{3}{2}X_5) + \bar{X}_3(X_3 - \tfrac{1}{2}X_4 + \tfrac{1}{2}X_5) \\ & + \bar{X}_4(-X_2 - \tfrac{1}{2}X_3) + \bar{X}_5(X_1 + \tfrac{3}{2}X_2 + \tfrac{1}{2}X_3) \\ = \{\bar{\mathbf{x}} X_5 + \mathbf{x}\bar{X}_5\} & + \{-\bar{X}_2 X_4 + \bar{X}_3(X_3 - \tfrac{1}{2}X_4) + \bar{X}_4(-X_2 - \tfrac{1}{2}X_3)\}, \end{aligned}$$

where $\quad \mathbf{x} \equiv X_1 + \tfrac{3}{2}X_2 + \tfrac{1}{2}X_3.$

Now

$$2(\bar{\mathbf{x}} X_5 + \mathbf{x}\bar{X}_5) \equiv (\mathbf{x} + X_5)(\bar{\mathbf{x}} + \bar{X}_5) - (\mathbf{x} - X_5)(\bar{\mathbf{x}} - \bar{X}_5)$$

and

$$-\bar{X}_2 X_4 + \bar{X}_3(X_3 - \tfrac{1}{2}X_4) + \bar{X}_4(-X_2 - \tfrac{1}{2}X_3) \text{ is } (i)^{r-1} f_1(X, X)$$

when $r = 3$ and X_1, X_2, X_3 are replaced by X_2, X_3, X_4.

Hence, if the statement is true when $r = 3$, it will be true when $r = 5$; so that we establish the result by induction.

Similarly, in the case of $f_1(x, \bar{y}) + f_1(\bar{x}, y)$ put (when $r = 5$)

$$\mathbf{x} \equiv x_1 + \tfrac{3}{2}x_2 + \tfrac{1}{2}x_3, \quad \mathbf{y} \equiv y_1 + \tfrac{3}{2}y_2 + \tfrac{1}{2}y_3.$$

Suppose that when the given Hermitian invariant of A is put in standard form it becomes

$$\xi_1\bar{\xi}_1+\ldots+\xi_k\bar{\xi}_k-\xi_{k+1}\bar{\xi}_{k+1}-\ldots-\xi_m\bar{\xi}_m,$$

where we may suppose without loss of generality that $2k \geq m$.

Then we have, exactly as in Ch. VI, § 12 :—

'A cannot have more than $2(m-k)$ characteristic-roots whose modulus is not unity. If exactly $2k-m$ characteristic-roots have unit modulus, the corresponding invariant-factors are linear';

and similarly for the other two theorems at the end of Ch. VI, substituting 'characteristic-root with unit modulus' for 'real characteristic-root'.*

Ex. 1. Find a substitution with invariant-factors $(\lambda-\alpha)$, $(\lambda-\bar{\alpha}^{-1})$, having $x\bar{x}-y\bar{y}$ as invariant.

[A substitution N with these invariant-factors is $x'=\alpha x$, $y'=\bar{\alpha}^{-1}y$, and it has as Hermitian invariant with non-zero determinant

$$\frac{i}{2}(x\bar{y}-\bar{x}y) \equiv X\bar{X}-Y\bar{Y}, \text{ where } x=X+iY,\ y=iX+Y$$

(see Ch. III, § 6). But when N is expressed in terms of X, Y, it becomes

$$X'=\frac{1}{2}(\alpha+\bar{\alpha}^{-1})X+\frac{i}{2}(\alpha-\bar{\alpha}^{-1})\,Y,$$

$$Y'=\frac{i}{2}(-\alpha+\bar{\alpha}^{-1})\,X+\frac{1}{2}(\alpha+\bar{\alpha}^{-1})\,Y,$$

which is the required substitution, when we replace X, Y by x, y.

This method enables us to find a substitution with Hermitian invariant $x_1\bar{x}_1+\ldots+x_k\bar{x}_k-x_{k+1}\bar{x}_{k+1}-\ldots-x_m\bar{x}_m$, and any given invariant-factors consistent with § 4.]

Ex. 2. Find a substitution with invariant-factor $(\lambda-e^{i\theta})^2$, and invariant $x\bar{x}-y\bar{y}$.

Ex. 3. If $A+E$ has non-zero determinant, where A is alternate, $2(A+E)^{-1}-E$ is orthogonal (a so-called 'Cayleyan' orthogonal substitution; see Cipolla, *Atti dell' Accad. Gioenia di Sci. nat. in Catania*, 5, VII, p. 1).

Every orthogonal substitution is the product of a Cayleyan substitution by a substitution of the type $(\pm x_1, \pm x_2, \ldots, \pm x_m)$.

What is the corresponding result for a substitution with invariant $x_1\bar{x}_1+\ldots+x_k\bar{x}_k-x_{k+1}\bar{x}_{k+1}-\ldots-x_m\bar{x}_m$?

* See Loewy, *Math. Annalen*, 1 (1898), p. 557.

CHAPTER IX

FAMILIES OF BILINEAR FORMS

§ 1. Family of Bilinear Forms.

THE bilinear forms

$$\Sigma a_{ij} y_i x_j - \lambda \Sigma b_{ij} y_i x_j \quad (i, j = 1, 2, \ldots, m)$$

obtained by varying λ are called a *family of bilinear forms.**

Suppose that by the substitution of

$$\left.\begin{array}{l} p_{t1}\mathbf{x}_1 + p_{t2}\mathbf{x}_2 + \ldots + p_{tm}\mathbf{x}_m \text{ for } x_m \\ q_{t1}\mathbf{y}_1 + q_{t2}\mathbf{y}_2 + \ldots + q_{tm}\mathbf{y}_m \text{ for } y_m \end{array}\right\} (t = 1, 2, \ldots, m)$$

$\Sigma a_{ij} y_i x_j$ becomes $\Sigma c_{ij}\mathbf{y}_i\mathbf{x}_j$, and $\Sigma b_{ij} y_i x_j$ becomes $\Sigma d_{ij}\mathbf{y}_i\mathbf{x}_j$.†

Then evidently the family

$$\Sigma a_{ij} y_i x_j - \lambda \Sigma b_{ij} y_i x_j \text{ becomes } \Sigma c_{ij}\mathbf{y}_i\mathbf{x}_j - \lambda \Sigma d_{ij}\mathbf{y}_i\mathbf{x}_j.$$

By a suitable choice of $\mathbf{x}_1, \ldots, \mathbf{x}_m, \mathbf{y}_1, \ldots, \mathbf{y}_m$ we can throw $\Sigma c_{ij}\mathbf{y}_i\mathbf{x}_j - \lambda \Sigma d_{ij}\mathbf{y}_i\mathbf{x}_j$ into a comparatively simple shape. This process will be given in § 2.

It has been proved in Ch. III, § 1, that the invariant-factors of the determinants of the forms

$$\Sigma a_{ij} y_i x_j - \lambda \Sigma b_{ij} y_i x_j \text{ and } \Sigma c_{ij} y_i x_j - \lambda \Sigma d_{ij} y_i x_j$$

are the same.

If we take

$$y_1 = x_1,\ y_2 = x_2,\ \ldots,\ y_m = x_m,\ a_{ij} = a_{ji},\ b_{ij} = b_{ji},$$

we obtain a *family of quadratic forms*.

Similarly, by taking

$$y_1 = \bar{x}_1,\ y_2 = \bar{x}_2,\ \ldots,\ y_m = \bar{x}_m,\ a_{ij} = \bar{a}_{ji},\ b_{ij} = \bar{b}_{ji},$$

we obtain a *family of Hermitian forms*.

We shall suppose in §§ 2, 3 that the determinant of the form $\Sigma b_{ij} y_i x_j$ does not vanish.

* *Eine Schaar von bilinearen Formen.*

† By Ch. III, § 1, $C = PAQ'$, $D = PBQ'$, with the usual notation in which A is the substitution corresponding to $\Sigma a_{ij} y_i x_j$, &c.

§ 2. Reduction of a Family of Bilinear Forms to its Canonical Shape.

The family of bilinear forms $\Sigma a_{ij} y_i x_j - \lambda \Sigma b_{ij} y_i x_j$ *can be reduced to the canonical shape*

$$a(\mathbf{x}, \mathbf{y}) - \lambda (\mathbf{x}_1 \mathbf{y}_1 + \mathbf{x}_2 \mathbf{y}_2 + \ldots + \mathbf{x}_m \mathbf{y}_m),$$

where $a(\mathbf{x}, \mathbf{y})$ *is the sum of bilinear forms of the type*

$$\mathbf{y}_1(\alpha \mathbf{x}_1 + \mathbf{x}_2) + \mathbf{y}_2 (\alpha \mathbf{x}_2 + \mathbf{x}_3) + \ldots + \mathbf{y}_{r-1}(\alpha \mathbf{x}_{r-1} + \mathbf{x}_r) + \mathbf{y}_r (\alpha \mathbf{x}_r),$$

provided the determinant of $\Sigma b_{ij} y_i x_j$ *does not vanish.*

Suppose variables $\xi_1, \xi_2, \ldots, \xi_m, \eta_1, \eta_2, \ldots, \eta_m$ chosen so that $\Sigma b_{ij} y_i x_j$ becomes

$$\xi_1 \eta_1 + \xi_2 \eta_2 + \ldots + \xi_m \eta_m$$

(see Ch. III, § 3).

Let $\Sigma a_{ij} y_i x_j$, when expressed in terms of these variables, become $\Sigma c_{ij} \eta_i \xi_j$. Then the family, when expressed in terms of the ξ's and the η's, becomes

$$\Sigma c_{ij} \eta_i \xi_j - \lambda (\xi_1 \eta_1 + \xi_2 \eta_2 + \ldots + \xi_m \eta_m). \qquad \text{(i)}$$

If the determinant of $\Sigma a_{ij} y_i x_j$ and therefore the determinant of $\Sigma c_{ij} \eta_i \xi_j$ is not zero, we put

$$\left.\begin{aligned} \xi_t &= p_{t1} \mathbf{x}_1 + p_{t2} \mathbf{x}_2 + \ldots + p_{tm} \mathbf{x}_m \\ \eta_t &= q_{t1} \mathbf{y}_1 + q_{t2} \mathbf{y}_2 + \ldots + q_{tm} \mathbf{y}_m \end{aligned}\right\} (t = 1, 2, \ldots, m),$$

where $Q' = P^{-1}$ and PCP^{-1} is a canonical substitution (Ch. I, § 9). Then the required transformation is performed, for by Ch. III, § 1,

$$\xi_1 \eta_1 + \xi_2 \eta_2 + \ldots + \xi_m \eta_m = \mathbf{x}_1 \mathbf{y}_1 + \mathbf{x}_2 \mathbf{y}_2 + \ldots + \mathbf{x}_m \mathbf{y}_m.$$

If the determinant of $\Sigma c_{ij} \eta_i \xi_j$ vanishes, we write (i) in the form

$$\{\Sigma c_{ij} \eta_i \xi_j + \epsilon (\xi_1 \eta_1 + \xi_2 \eta_2 + \ldots + \xi_m \eta_m)\} \\ - (\lambda + \epsilon) (\xi_1 \eta_1 + \xi_2 \eta_2 + \ldots + \xi_m \eta_m)$$

and take P^{-1} as the substitution transforming $C + \epsilon E$ into canonical shape. (ϵ is chosen so that the determinant of $C + \epsilon E$ does not vanish.)

Then we reduce

$$\Sigma c_{ij} \eta_i \xi_j + \epsilon (\xi_1 \eta_1 + \xi_2 \eta_2 + \ldots + \xi_m \eta_m)$$

to the sum of bilinear forms of the type

$$\mathbf{y}_1 ((\alpha + \epsilon) \mathbf{x}_1 + \mathbf{x}_2) + \ldots + \mathbf{y}_{r-1} ((\alpha + \epsilon) \mathbf{x}_{r-1} + \mathbf{x}_r) + \mathbf{y}_r ((\alpha + \epsilon) \mathbf{x}_r)$$

and also reduce

$$(\lambda+\epsilon)(\xi_1\eta_1+\xi_2\eta_2+\ldots+\xi_m\eta_m)$$

to $$(\lambda+\epsilon)(\mathbf{x}_1\mathbf{y}_1+\mathbf{x}_2\mathbf{y}_2+\ldots+\mathbf{x}_m\mathbf{y}_m),$$

which completes the required transformation.

Since transformation of the variables does not alter the invariant-factors of the determinant of a family of bilinear forms, the invariant-factors of the determinant of

$$\Sigma a_{ij}y_ix_j-\lambda\Sigma b_{ij}y_ix_j$$

are $(\lambda-\alpha)^r$, &c.

Conversely:—

If the determinants of two families of bilinear forms

$$\Sigma a_{ij}y_ix_j-\lambda\Sigma b_{ij}y_ix_j \quad \textit{and} \quad \Sigma c_{ij}y_ix_j-\lambda\Sigma d_{ij}y_ix_j$$

have the same invariant-factors, and the determinants of

$$\Sigma b_{ij}y_ix_j \quad \textit{and} \quad \Sigma d_{ij}y_ix_j$$

do not vanish, one family can be transformed into the other by a suitable change of variables.

For both families can be transformed into the same family, which is the sum of forms of the type

$$\{y_1(\alpha x_1+x_2)+\ldots+y_{r-1}(\alpha x_{r-1}+x_r)+y_r(\alpha x_r)\}$$
$$-\lambda\{x_1y_1+x_2y_2+\ldots+x_ry_r\}.$$

Corollary.

The family of bilinear forms can be transformed into the sum of families of the type

$$\{(\alpha-\lambda)(y_1x_r+y_2x_{r-1}+y_3x_{r-2}+\ldots+y_rx_1)$$
$$+(y_2x_r+y_3x_{r-1}+\ldots+y_rx_2)\}.$$

For the determinant of this family has invariant-factors $(\lambda-\alpha)^r$, &c., as is readily verified.

Ex. 1. Transform

$$\{y_1(4x_1-x_2+x_3)+y_2(9x_1+8x_2+7x_3)+y_3(7x_1+x_2+3x_3)\}$$
$$-\lambda\{y_1(-2x_2-x_3)+y_2(2x_1+3x_2+2x_3)+y_3(-x_1-x_2-x_3)\}$$

into canonical shape.

[Put $\xi_1=-2x_2-x_3$, $\xi_2=2x_1+3x_2+2x_3$, $\xi_3=-x_1-x_2-x_3$. The family becomes

$$\{y_1(3\xi_1+\xi_2-2\xi_3)+y_2(2\xi_1+3\xi_2-3\xi_3)$$
$$+y_3(4\xi_1+2\xi_2-3\xi_3)\}-\lambda\{y_1\xi_1+y_2\xi_2+y_3\xi_3\}.$$

Now $(3x_1+x_2-2x_3,\ 2x_1+3x_2-3x_3,\ 4x_1+2x_2-3x_3)$
is transformed into $(x_1+x_2,\ x_2+x_3,\ x_3)$
by $(x_1,\ 2x_1+x_2-2x_3,\ -2x_1+x_3)$,
whose inverse is $(x_1,\ 2x_1+x_2+2x_3,\ 2x_1+x_3)$.

We put then

$$\left.\begin{array}{l}\xi_1=\mathbf{x}_1,\ \xi_2=2\mathbf{x}_1+\mathbf{x}_2+2\mathbf{x}_3,\ \xi_3=2\mathbf{x}_1+\mathbf{x}_3\\ y_1=\mathbf{y}_1+2\mathbf{y}_2-2\mathbf{y}_3,\ y_2=\mathbf{y}_2,\ y_3=-2\mathbf{y}_2+\mathbf{y}_3\end{array}\right\},$$

and the family takes the canonical shape

$$\{\mathbf{y}_1(\mathbf{x}_1+\mathbf{x}_2)+\mathbf{y}_2(\mathbf{x}_2+\mathbf{x}_3)+\mathbf{y}_3\}-\lambda\{\mathbf{x}_1\mathbf{y}_1+\mathbf{x}_2\mathbf{y}_2+\mathbf{x}_3\mathbf{y}_3\}.]$$

Ex. 2. Transform into canonical shape

$$\{-4y_1x_1+y_2(2x_1-x_2)\}-\lambda\{y_1(-2x_1+x_2)+y_2(x_1-x_2)\},$$

and $\{y_1(-2x_2-2x_3)+y_2(2x_1-5x_2-2x_3)+y_3(x_1-3x_2-2x_3)\}$

$$-\lambda\{y_1(2x_2-x_3)+y_2(-2x_1+3x_2-x_3)+y_3(-x_1+2x_2-x_3)\}.$$

§ 3. Reduction of a Family of Quadratic Forms to its Canonical Shape.

The family of quadratic forms

$$\Sigma a_{ij}x_ix_j-\lambda\Sigma b_{ij}x_ix_j,$$

where $a_{ij}=a_{ji}$ and $b_{ij}=b_{ji}$, can be reduced to the sum σ of families of quadratic forms of the type

$$\{(\alpha-\lambda)(\mathbf{x}_1\mathbf{x}_r+\mathbf{x}_2\mathbf{x}_{r-1}+\mathbf{x}_3\mathbf{x}_{r-2}+\ldots+\mathbf{x}_r\mathbf{x}_1)+(\mathbf{x}_2\mathbf{x}_r+\mathbf{x}_3\mathbf{x}_{r-1}+\ldots+\mathbf{x}_r\mathbf{x}_2)\},$$

provided the determinant of $\Sigma b_{ij}x_ix_j$ does not vanish.

Let $\xi_1,\ \xi_2,\ \ldots,\ \xi_r$ be independent linear functions of $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_r$ such that

$$\mathbf{x}_1\mathbf{x}_r+\mathbf{x}_2\mathbf{x}_{r-1}+\ldots+\mathbf{x}_r\mathbf{x}_1\equiv\xi_1^2+\xi_2^2+\ldots+\xi_r^2,$$

and similarly for each constituent of the sum σ.

For instance, we might take

$$\sqrt{2}\,\xi_1=\mathbf{x}_1+\mathbf{x}_r,\ \sqrt{2}\,\xi_r=i(\mathbf{x}_1-\mathbf{x}_r),\ \sqrt{2}\,\xi_2=\mathbf{x}_2+\mathbf{x}_{r-1},\ \sqrt{2}\,\xi_{r-1}=i(\mathbf{x}_2-\mathbf{x}_{r-1}),\ \&c.$$

Suppose that this reduces the sum of forms such as

$$\alpha(\mathbf{x}_1\mathbf{x}_r+\mathbf{x}_2\mathbf{x}_{r-1}+\ldots+\mathbf{x}_r\mathbf{x}_1)+(\mathbf{x}_2\mathbf{x}_r+\mathbf{x}_3\mathbf{x}_{r-1}+\ldots+\mathbf{x}_r\mathbf{x}_2)$$

to $\Sigma k_{ij}\xi_i\xi_j$ and that K is the corresponding symmetric substitution.

The invariant-factors of the determinant of σ are readily seen to be $(\lambda-\alpha)^r$, &c., and therefore the determinant of

$$\Sigma k_{ij}\xi_i\xi_j-\lambda(\xi_1^2+\xi_2^2+\xi_3^2+\ldots)$$

has the invariant-factors $(\lambda-\alpha)^r$, &c., which may be supposed the same as the invariant-factors of the determinant of the given family, since α and r are at our disposal.

Choose now independent linear functions $X_1, X_2, \ldots, X_m$ of $x_1, x_2, \ldots, x_m$ so that

$$\Sigma b_{ij}x_ix_j \equiv X_1^2+X_2^2+\ldots+X_m^2.$$

Suppose $\Sigma a_{ij}x_ix_j \equiv \Sigma c_{ij}X_iX_j$, and let A, C be the symmetric substitutions corresponding to these two forms.

Take now the orthogonal substitution P such that $PCP^{-1}=K$ (Ch. VI, § 5).

Since $P^{-1}=P'$, the transformation

$$p_{t1}\xi_1+p_{t2}\xi_2+\ldots+p_{tm}\xi_m \text{ for } X_t$$

transforms $\quad \Sigma c_{ij}X_iX_j$ into $\Sigma k_{ij}\xi_i\xi_j$

and $\quad X_1^2+X_2^2+\ldots+X_m^2$ into $\xi_1^2+\xi_2^2+\ldots+\xi_m^2$.

Hence the family $\Sigma a_{ij}x_ix_j-\lambda\Sigma b_{ij}x_ix_j$ may be transformed into $\Sigma k_{ij}\xi_i\xi_j-\lambda(\xi_1^2+\xi_2^2+\ldots+\xi_m^2)$, which is transformable into the given canonical shape.

We have assumed that the determinant of A does vanish. If it does, we proceed as in § 2.

As in § 2, we have :—

If the determinants of the two families of quadratic forms

$$\Sigma a_{ij}x_ix_j-\lambda\Sigma b_{ij}x_ix_j \text{ and } \Sigma c_{ij}x_ix_j-\lambda\Sigma d_{ij}x_ix_j$$

(where $a_{ij}=a_{ji}$, $b_{ij}=b_{ji}$, $c_{ij}=c_{ji}$, $d_{ij}=d_{ji}$) have the same invariant-factors, and the determinants of $\Sigma b_{ij}x_ix_j$ and $\Sigma d_{ij}x_ix_j$ do not vanish, one family can be transformed into the other by a suitable change of variables.

Ex. 1. Transform into canonical shape the family

$$(2x^2+64xy-113y^2)-\lambda(25x^2-100xy+125y^2).$$

[Put $5X=x+2y$, $5Y=y$. Then the family becomes

$$\tfrac{1}{25}(2X^2+72XY+23Y^2)-\lambda(X^2+Y^2).$$

Now the symmetric substitution $\left(\frac{2x+36y}{25}, \frac{36x+23y}{25}\right)$ is

transformed into $(2x, \; -y)$ by the orthogonal substitution $\left(\frac{3x+4y}{5}, \; \frac{-4x+3y}{5}\right)$ whose inverse is $\left(\frac{3x-4y}{5}, \; \frac{4x+3y}{5}\right)$. Put therefore $5X = 3\mathbf{x}-4\mathbf{y}$, $5Y = 4\mathbf{x}+3\mathbf{y}$, and the family becomes $(2\mathbf{x}^2-\mathbf{y}^2)-\lambda(\mathbf{x}^2+\mathbf{y}^2)$.]

Ex. 2. Transform into canonical shape the families

$$(5x^2-6xy+y^2)-\lambda(2x^2-2xy),$$
$$(-2x^2+y^2+2yz-2xz)-\lambda(2x^2+y^2+2xz+2xy),$$
$$(2z^2-4yz+4zx-4xy)-\lambda(3x^2+3y^2+z^2-2yz+2zx-2xy).$$

Ex. 3. The equation $S-\lambda S' = 0$, where

$$S \equiv ax^2+2hxy+by^2 \text{ and } S' \equiv a'x^2+2h'xy+b'y^2,$$

represents an involution pencil.

If $S-\lambda S'$, when put in canonical shape, becomes

$$(\lambda_1\mathbf{x}^2+\lambda_2\mathbf{y}^2)-\lambda(\mathbf{x}^2+\mathbf{y}^2),$$

$\mathbf{x} = 0$, $\mathbf{y} = 0$ are the double lines of the pencil.

Putting $y = 1$, evaluate $\int \frac{dx}{S\sqrt{S'}}$ by the substitution $z = \mathbf{x}/\mathbf{y}$.

For instance, take $S \equiv 3x^2-4x+2$, $S' \equiv 2x^2-2x+1$.

[$\mathbf{x}$, $\mathbf{y}$ are very readily obtained; for $(\lambda_1-\lambda_2)\mathbf{x}^2 = S-\lambda_2 S'$, &c. The integral is made to depend on

$$\int \frac{z\,dz}{(\lambda_1 z^2+\lambda_2)(1+z^2)^{\frac{1}{2}}} \text{ and } \int \frac{dz}{(\lambda_1 z^2+\lambda_2)(1+z^2)^{\frac{1}{2}}}.$$

Now put $1+z^2 = u^2$ or $1/(1-v^2)$, respectively.]

Ex. 4. Verify by means of this section that the family $S-\lambda S'$, where

$$S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy,$$
$$S' \equiv a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy,$$

and S' is non-degenerate, can be reduced to one of the shapes $\sigma-\lambda\sigma'$ given in the first table of Ch. IV, § 9; and similarly for conicoids.

Ex. 5. Prove that the family of Hermitian forms

$$\Sigma a_{ij}\bar{x}_i x_j - \lambda \Sigma b_{ij}\bar{x}_i x_j,$$

where $a_{ij} = \bar{a}_{ji}$ and $b_{ij} = \bar{b}_{ji}$, can be transformed into

$$(\lambda_1\mathbf{x}_1\bar{\mathbf{x}}_1+\lambda_2\mathbf{x}_2\bar{\mathbf{x}}_2+\dots+\lambda_m\mathbf{x}_m\bar{\mathbf{x}}_m)-\lambda(\mathbf{x}_1\bar{\mathbf{x}}_1+\mathbf{x}_2\bar{\mathbf{x}}_2+\dots+\mathbf{x}_m\bar{\mathbf{x}}_m)$$

if $\Sigma b_{ij}\bar{x}_i x_j$ is a *positive* Hermitian form.

Ex. 6. If $\Sigma a_{ij}x_i x_j$ and $\Sigma b_{ij}x_i x_j$ are real quadratic forms, the latter being *positive*, $\Sigma a_{ij}x_i x_j - \lambda \Sigma b_{ij}x_i x_j$ can be transformed into

$$(\lambda_1\mathbf{x}_1{}^2+\lambda_2\mathbf{x}_2{}^2+\dots+\lambda_m\mathbf{x}_m{}^2)-\lambda(\mathbf{x}_1{}^2+\mathbf{x}_2{}^2+\dots+\mathbf{x}_m{}^2)$$

by a real transformation.

[By a real transformation we can transform the given family into $\Sigma c_{ij}\xi_i\xi_j - \lambda(\xi_1^2+\xi_2^2+\ldots+\xi_m^2)$. Now put

$$\xi_t = p_{t1}\mathbf{x}_1 + p_{t2}\mathbf{x}_2 + \ldots + p_{tm}\mathbf{x}_m,$$

where P^{-1} is the real orthogonal substitution transforming C into a multiplication (Ch. I, § 12).]

Ex. 7. If the determinants of the alternate form $\Sigma a_{ij}y_ix_j$ and the symmetric form $\Sigma b_{ij}y_ix_j$ do not vanish, the family

$$\Sigma a_{ij}y_ix_j - \lambda\Sigma b_{ij}y_ix_j$$

can be transformed into the sum of families of the type

$$(\alpha-\lambda)(\mathbf{y}_{2r}\mathbf{x}_1+\mathbf{y}_{2r-1}\mathbf{x}_2+\ldots+\mathbf{y}_{r+1}\mathbf{x}_r)$$
$$+(-\alpha-\lambda)(\mathbf{y}_r\mathbf{x}_{r+1}+\mathbf{y}_{r-1}\mathbf{x}_{r+2}+\ldots+\mathbf{y}_1\mathbf{x}_{2r})$$
$$+(\mathbf{y}_{2r}\mathbf{x}_2+\mathbf{y}_{2r-1}\mathbf{x}_3+\ldots+\mathbf{y}_{r+2}\mathbf{x}_r)-(\mathbf{y}_r\mathbf{x}_{r+2}+\mathbf{y}_{r-1}\mathbf{x}_{r+3}+\ldots+\mathbf{y}_2\mathbf{x}_{2r}).$$

[As in § 3, using the last theorem of Ch. VI, § 9.]

§ 4.

We may also prove the result at the end of § 3 by the aid of Ch. VI, § 4, Corollary III, without using the results of Ch. VI, §§ 5 and 6.

We know by § 2 that the bilinear family

$$\Sigma\, a_{ij}y_ix_j - \lambda\Sigma\, b_{ij}y_ix_j$$

can be transformed (as in § 1) into

$$\Sigma\, c_{ij}y_ix_j - \lambda\Sigma\, d_{ij}y_ix_j,$$

since the determinants of the two families have the same invariant-factors. Then, with the notation of § 1, we have $PAQ' = C$ and $PBQ' = D$.

If, then, we can find R such that $RAR' = C$ and $RBR' = D$, the quadratic family $\Sigma\, a_{ij}x_ix_j - \lambda\Sigma\, b_{ij}x_ix_j$ will be transformed into $\Sigma\, c_{ij}x_ix_j - \lambda\Sigma\, d_{ij}x_ix_j$ on replacing

$$x_t \text{ by } r_{t1}x_1 + r_{t2}x_2 + \ldots + r_{tm}x_m \quad (t = 1, 2, \ldots, m)$$

on the supposition $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$, $c_{ij} = c_{ji}$, $d_{ij} = d_{ji}$.

Now since QAQ' and $PQ^{-1}.QAQ' = C$ are symmetric, by Ch. VI, § 4, Corollary III we can find T, depending solely on PQ^{-1} (not on QAQ' or A), such that

$$T.QAQ'.T' = PQ^{-1}.QAQ'.$$

Then, since T does not involve A, we shall have also

$$T.QBQ'.T' = PQ^{-1}.QBQ'.$$

Putting $TQ = R$, we have $RAR' = C$ and $RBR' = D$, as required.

Ex. 1. Let K, K_1, L, L_1 be the substitutions corresponding to the bilinear forms

$$k \equiv \{y_1(-4x_1-2x_2+3x_3)+y_2(-2x_1+x_2)+y_3(3x_1-x_3)\},$$
$$k_1 \equiv \{y_1(-4x_1-4x_2+5x_3)+y_2(-4x_1+x_2+x_3)+y_3(5x_1+x_2-3x_3)\},$$
$$l \equiv \{y_1(-4x_1-4x_2+5x_3)+y_2(-4x_1+x_2+x_3)+y_3(5x_1+x_2-3x_3)\},$$
$$l_1 \equiv \{y_1(-6x_2+5x_3)+y_2(-6x_1+x_2+2x_3)+y_3(5x_1+2x_2-4x_3)\}.$$

Then the family of bilinear forms $k-\lambda k_1$ is transformed into $l-\lambda l_1$ when we replace

$$x_1 \text{ by } 3x_1+x_2-2x_3,\quad x_2 \text{ by } 2x_1+3x_2-3x_3,\quad x_3 \text{ by } 4x_1+2x_2-3x_3.$$

Hence, if $A \equiv (3x_1+x_2-2x_3,\ 2x_1+3x_2-3x_3,\ 4x_1+2x_2-3x_3)$, $AK = L$ and $AK_1 = L_1$.

Now, by Ch. VI, § 4, Ex. 2, $RKR' = L$, $RK_1R' = L_1$, where

$$R \equiv (\tfrac{9}{4}x_1+\tfrac{1}{2}x_2-\tfrac{9}{8}x_3,\ \tfrac{3}{2}x_1+2x_2-\tfrac{7}{4}x_3,\ \tfrac{5}{2}x_1+x_2-\tfrac{5}{4}x_3).$$

Hence the family $k-\lambda k_1$ is transformed into $l-\lambda l_1$ when we perform on it the *congruent* transformation corresponding to R.

Ex. 2. If $\Sigma a_{ij}y_ix_j$ and $\Sigma b_{ij}y_ix_j$ are alternate bilinear forms with non-zero determinant, the invariant-factors of the determinant of $\Sigma a_{ij}y_ix_j-\lambda\Sigma b_{ij}y_ix_j$ occur in pairs of the type

$$(\lambda-\alpha)^r,\ (\lambda-\alpha)^r.$$

[Transform $\Sigma b_{ij}y_ix_j$ by a congruent transformation into

$$(x_1y_2-x_2y_1)+(x_3y_4-x_4y_3)+\dots,$$

as in Ch. III, § 7. Then the determinant of the family is of the type

$$\begin{vmatrix} 0 & a+\lambda & b & c \\ -a-\lambda & 0 & d & e \\ -b & -d & 0 & k+\lambda \\ -c & -e & -k-\lambda & 0 \end{vmatrix}$$

$$\text{or}\quad \begin{vmatrix} -a-\lambda & 0 & d & e \\ 0 & -a-\lambda & -b & -c \\ -c & -e & -k-\lambda & 0 \\ b & d & 0 & -k-\lambda \end{vmatrix},$$

taking the forms of degree 4 as an illustration.

But the latter determinant is the characteristic-determinant of the product of the two alternate substitutions with matrices

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & k \\ -c & -e & -k & 0 \end{vmatrix} \quad\text{and}\quad \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

Now use Ch. VI, § 8.]

Ex. 3. Show that the family of alternate bilinear forms

$$\Sigma a_{ij} y_i x_j - \lambda \Sigma b_{ij} y_i x_j$$

of Ex. 2 can be transformed into the sum of families of the type

$$(-\alpha+\lambda)(\mathbf{y}_{2r}\mathbf{x}_1+\mathbf{y}_{2r-1}\mathbf{x}_2+\dots+\mathbf{y}_{r+1}\mathbf{x}_r)$$
$$+(\alpha-\lambda)(\mathbf{y}_r\mathbf{x}_{r+1}+\mathbf{y}_{r-1}\mathbf{x}_{r+2}+\dots+\mathbf{y}_1\mathbf{x}_{2r})$$
$$-(\mathbf{y}_{2r}\mathbf{x}_2+\mathbf{y}_{2r-1}\mathbf{x}_3+\dots+\mathbf{y}_{r+2}\mathbf{x}_r)+(\mathbf{y}_r\mathbf{x}_{r+2}+\mathbf{y}_{r-1}\mathbf{x}_{r+3}+\dots+\mathbf{y}_2\mathbf{x}_{2r}),$$

and that any such family can be transformed into any given similar family with the same invariant-factors.

[Use the method of § 4, and Ch. VI, § 8, Ex. 6.]

Ex. 4. Illustrate the theorem of Ex. 3 by a simple example.

[Use Ch. VI, § 8, Ex. 7 in the same manner that Ch. VI, § 4, Ex. 2 was used in Ex. 1.]

Ex. 5. If $\Sigma a_{ij}\bar{x}_i x_j$ and $\Sigma b_{ij}\bar{x}_i x_j$ are Hermitian forms with non-zero determinant, the unreal invariant-factors of the determinant of $\Sigma a_{ij}\bar{x}_i x_j - \lambda \Sigma b_{ij}\bar{x}_i x_j$ occur in pairs of the type

$$(\lambda-\alpha)^r,\ (\lambda-\bar{\alpha})^r.$$

[Transform $\Sigma b_{ij}\bar{x}_i x_j$ into

$$x_1\bar{x}_1+x_2\bar{x}_2+\dots+x_k\bar{x}_k-x_{k+1}\bar{x}_{k+1}-\dots-x_m\bar{x}_m.$$

Then the determinant of the family becomes the characteristic-determinant of a substitution of the type discussed in Ch. VI, § 12.]

Ex. 6. Show that the family of Hermitian forms

$$\Sigma a_{ij}\bar{x}_i x_j - \lambda \Sigma b_{ij}\bar{x}_i x_j$$

of Ex. 5 can be transformed into the sum of families of the type

$$(\alpha-\lambda)(\bar{\mathbf{x}}_{2r}\mathbf{x}_1+\bar{\mathbf{x}}_{2r-1}\mathbf{x}_2+\dots+\bar{\mathbf{x}}_{r+1}\mathbf{x}_r)$$
$$+(\bar{\alpha}-\lambda)(\bar{\mathbf{x}}_r\mathbf{x}_{r+1}+\bar{\mathbf{x}}_{r-1}\mathbf{x}_{r+2}+\dots+\bar{\mathbf{x}}_1\mathbf{x}_{2r})$$
$$+(\bar{\mathbf{x}}_{2r}\mathbf{x}_2+\bar{\mathbf{x}}_{2r-1}\mathbf{x}_3+\dots+\bar{\mathbf{x}}_{r+2}\mathbf{x}_r)+(\bar{\mathbf{x}}_r\mathbf{x}_{r+2}+\bar{\mathbf{x}}_{r-1}\mathbf{x}_{r+3}+\dots+\bar{\mathbf{x}}_2\mathbf{x}_{2r})$$

where α is unreal, and

$$(\alpha-\lambda)(\bar{\mathbf{x}}_r\mathbf{x}_1+\bar{\mathbf{x}}_{r-1}\mathbf{x}_2+\dots+\bar{\mathbf{x}}_1\mathbf{x}_r)+(\bar{\mathbf{x}}_r\mathbf{x}_2+\bar{\mathbf{x}}_{r-1}\mathbf{x}_3+\dots+\bar{\mathbf{x}}_2\mathbf{x}_r)$$

where α is real, and that any such family can be transformed into any given similar family with the same invariant-factors.

[As in Ex. 3, using Ch. VI, § 11, Ex. 1.]

§ 5. A Family in which both Determinants vanish.

If in § 2 the determinants of both $\Sigma a_{ij} y_i x_j$ and $b_{ij} y_i x_j$ vanish, we may reduce the family of bilinear forms to a simpler shape as follows.

Take any quantity ϵ such that the determinant of

$$\Sigma \mathbf{b}_{ij} y_i x_j \equiv \Sigma b_{ij} y_i x_j - \epsilon \Sigma a_{ij} y_i x_j$$

does not vanish.

Then the family of bilinear forms becomes

$$(1 - \lambda\epsilon) \{\Sigma a_{ij} y_i x_j - \boldsymbol{\lambda} \Sigma \mathbf{b}_{ij} y_i x_j\},$$

where $(1 - \lambda\epsilon) \boldsymbol{\lambda} = \lambda$.

This family may be transformed into simpler shape as in § 2; and then we replace $\boldsymbol{\lambda}$ by $\lambda/(1 - \lambda\epsilon)$.

A similar method applies to a family of quadratic forms.

For further details of this method, and for the case in which the determinant of the family vanishes identically, we must refer to the treatises quoted in the preface.

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